Inscribed Square Problem

Submitted by:

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Abstract

There have still been many mathematical problems still yet to be solved, and some yet to have space to be extended and further pursued. Our group has decided to take the Inscribed Square Problem and challenge ourselves to extend it on a larger scale with shapes such as the heart and ellipse.

We brought in theories such as Immediate Value Theorem and Inscribed Rectangle to aid us in proving that there can be an Inscribed Square in the figure. Then we used the proven possible figures to calculate the area of the Jordan curve.

From that, we plotted a graph to come up with equations for each Jordan curve and tried to find a pattern that linked all the Jordan Curves, even though there were some limitations that prevented us from finding an ideal pattern.

Acknowledgement

The students involved in this project would like to thank the school for the opportunity to participate in this competition.

They are also very grateful to the project supervisor, Ms. Kok Lai Fong, for her guidance in the course of preparing this project.

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Chapter 1 – Introduction

1.1 Objective

The aim of our project is to find the relationship between the area of the inscribed square in a Jordan curve and the area inside the Jordan curve. In this project the Jordan curves we used are isosceles, equilateral and right-angled triangles, square, circle, ellipse and heart.

1.2 Problem

However, there has not been a specific proof for the inscribed square problem for triangles, heart, circle, square and ellipse. Thus, we have decided to make use of existing theorems to prove that a square can be inscribed in these various shapes. Furthermore, we derived the area of the inscribed square and area of the Jordan curve surrounding it.

1.3 Background

The inscribed square problem, also known as the square peg problem or the Toeplitz's conjecture is an unsolved problem which has an objective to find out if there can be a square inscribed in every single Jordan curve.

A Jordan curve is a plane simple closed curve, which is a non-self-intersecting continuous loop in the plane, separates the plane into two regions, inside of the curve and outside. It does not necessarily have to be a curve (e.g. polygons). The four vertices of the square have to touch the curve for it to be inscribed.

The problem was proposed by Otto Toeplitz in 1911 and many people have attempted to solve it yet few have attempted to find the area of the square inscribed in the curves. As of 2017, the case has remained open. Therefore, our project researches on the area of the inscribed square and the area of the Jordan curve around it.

Chapter 2 - Literature Review

2.1 Overview

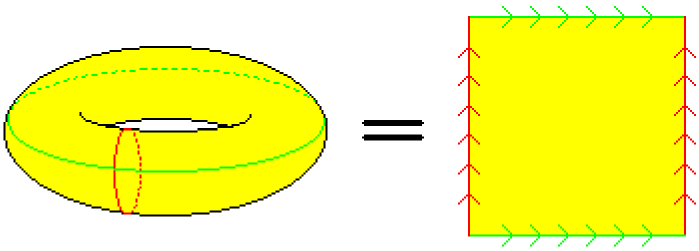
We will prove that there can be an inscribed square in some Jordan curves using the inscribed rectangle theorem and intermediate value theorem.

We will also be using the common formulas to calculate the area of the inscribed squares and the areas of the Jordan curves (isosceles, equilateral and right-angled triangles, square, circle, and ellipse).

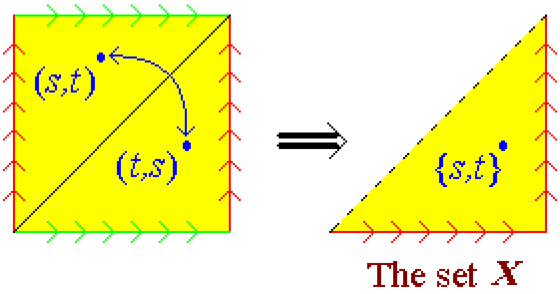
2.2 Inscribed Rectangle Theorem

The Inscribed Rectangle Theorem proves that every simple closed curve has at least one inscribed rectangle.

Now, we will prove the Inscribed Rectangle Theorem. Firstly, since the simple closed curve J is topologically the same as the circle C, the Cartesian product $J \times J$ is topologically equivalent to $C \times C$, which is a torus, that is a three-dimensional, doughnut-shaped solid as shown in the first picture in the figure below. Let us imagine $C \times C$ as a union of circles all placed around another circle, giving the surface of a torus.



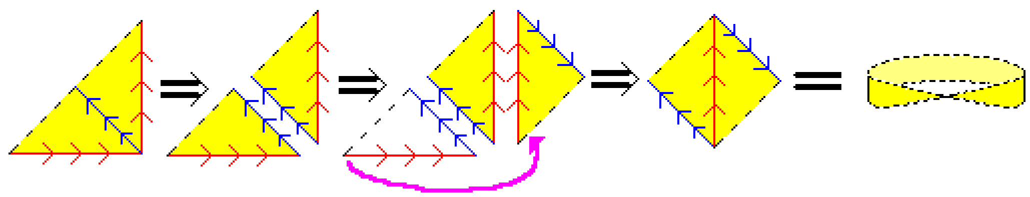
To construct a torus, we draw the square $[0,1] \times [0,1]$, with the understanding that the pairs of opposite edges are to be connected together so that $(0,2/3)$ and $(1,2/3)$ actually represent the same point of the torus as shown in the figure above.

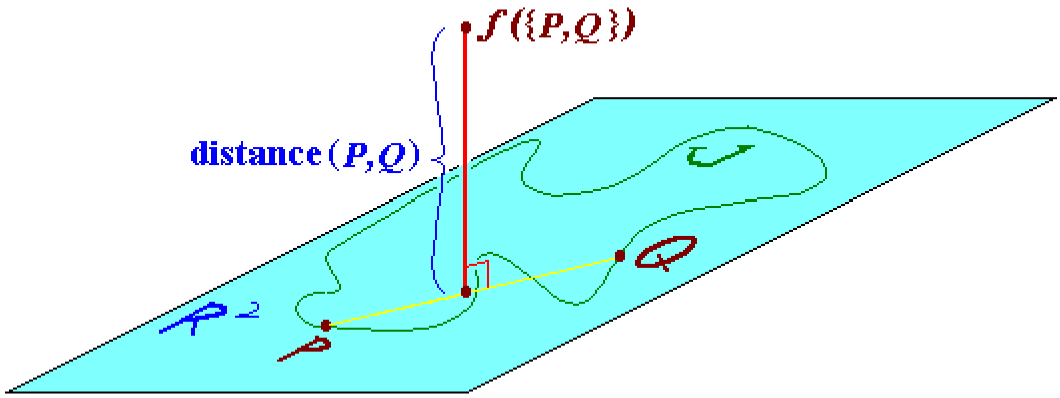


Let X represent the set of all unordered pairs of points on the curve J. In other words, a "point" in X will be a pair $\{P,Q\}$ of points on J. Since the torus is $J \times J$, the points on the torus represent ordered pairs of points on J. However, since both (P,Q) and (Q,P) in $J \times J$ represent the "point" $\{P,Q\}$ in X, thus topologically, X is the result when we "fold" the torus over onto itself, matching up (P,Q) and (Q,P).

By representing the torus as the square $[0,1] \times [0,1]$, we get a picture of X as the result when we fold this square across its diagonal, so as to identify the points (s,t) and (t,s) into a single "point" $\{s,t\}$. We notice that the diagonal of the square would correspond to ordered pairs (P,P), and thus it would also correspond to one-element sets $\{P\}$ of points on J, instead of two-element sets like the ones in X.

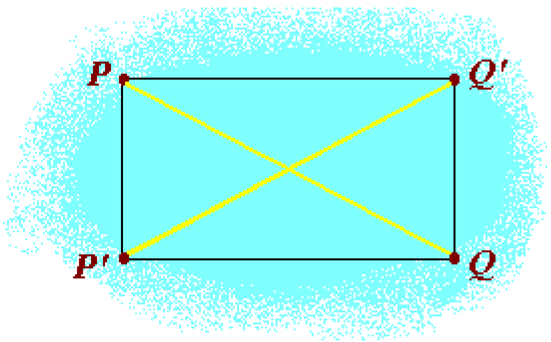
By imagining X as a piece of paper, whose shape we modify by "cutting it up into pieces" and "pasting the pieces back together" differently, we obtain the result that X is a Mobius strip, specifically, an open Mobius strip as its bounding curve is the missing diagonal of "singleton sets" as shown in the figure below. Note that it is a Mobius strip because in the last step, we only connect it after aligning the blue arrows on the opposite edges, which requires a twist.





Next, imagine R^2 as the plane z=0 sitting inside x,y,z-space R^3, and imagine J lying on that plane. Define a function $f:X$ ---> R^3 by the following rule: $f(\{P,Q\})$ is the point in R^3 lying directly over the midpoint of the segment PQ but with the z-coordinate equal to the distance from P to Q. Clearly, this function takes the Mobius strip X into the region above the x,y-plane, and connects its boundary onto the curve J. This is because as the "points" $\{P,Q\}$ of X get close to the boundary of X, the distance from P to Q becomes very small and thus the z-coordinate of $f(\{P,Q\})$ approaches zero.

It is impossible to do this in a one-to-one way, that is, the image set $f(X)$ must have some collisions where more than one point of X is carried to the same point of R^3. We cannot embed the space we get by connecting a disk to a Mobius strip, called the "projective plane", into three dimensions.



Now, we know that there must be two pairs of points $\{P,Q\}$ and $\{P',Q'\}$ such that $f(\{P,Q\}) = f(\{P',Q'\})$. Thus, this means that the segments PQ and $P'Q'$ meet at their midpoints and are of the same length. Therefore, from this we can conclude that $PP'QQ'$ is a rectangle.

2.3 Intermediate Value Theorem

The Intermediate Value Theorem (IVT) states that if a continuous function, f, with an interval, [a, b], as its domain, takes values f(a) and f(b) at each end of the interval, then it also takes any value between f(a) and f(b) at some point within the interval.

The IVT also states that if a continuous function takes on two values y_1 and y_2 at points a and b, it also takes on every value between y_1 and y_2 at some point between a and b.

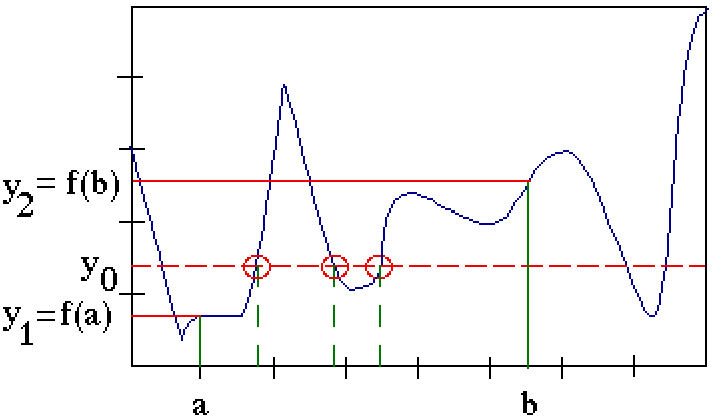
For example, if $e^0 = 1$ and $e^1 = e$,

$$1 < 2 < e,$$

$$e^x \text{ is continuous for all real } x$$

Thus, we can conclude that $e^x = 2$ for some x with $0 \leq x \leq 1$.

Graphically, the IVT states that if $y_1 = f(a)$ and $y_2 = f(b)$ for a function f(x), and if we draw the horizontal line $y = y_0$ for any y_0 between y_1 and y_2, then the horizontal line will intersect the graph of $y = f(x)$ at at least one point whose x-coordinate is between a and b.



The formal statement of the IVT is as follows:

Let f(x) be a function which is continuous on the closed interval [a,b] and let y_0 be a real number lying between f(a) and f(b), i.e. with $f(a) \leq y_0 \leq f(b)$ or $f(b) \leq y_0 \leq f(a)$.

There will be at least one c with $a \leq c \leq b$ such that $y_0 = f(c)$.

Note that "continuous on the closed interval [a,b]" means that f(x) is continuous at every point x with $a < x < b$ and that f(x) is right-continuous at $x = a$ and left-continuous at $x = b$.

2.4 Area of Jordan curves

We used the formulas below to calculate the area of the Jordan curves used in this project, that are the isosceles, equilateral and right-angled triangles, square, circle, and ellipse.

$A =$ Area

1. Triangle:

$$A = \frac{1}{2}bh$$

where $b =$ base, $h =$ height

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

where a, b, and c are the lengths of the sides and $s = \frac{1}{2}(a+b+c)$ or half the perimeter

2. Equilateral triangle

$$A = \frac{s^2\sqrt{3}}{4}$$

where $s =$ side

3. Square

$$A = s^2$$

where $s =$side

4. Circle

$$A = \pi r^2$$

where $r =$ radius

5. Ellipse

$$A = \pi ab$$

where $a =$ major axis, $b =$ minor axis

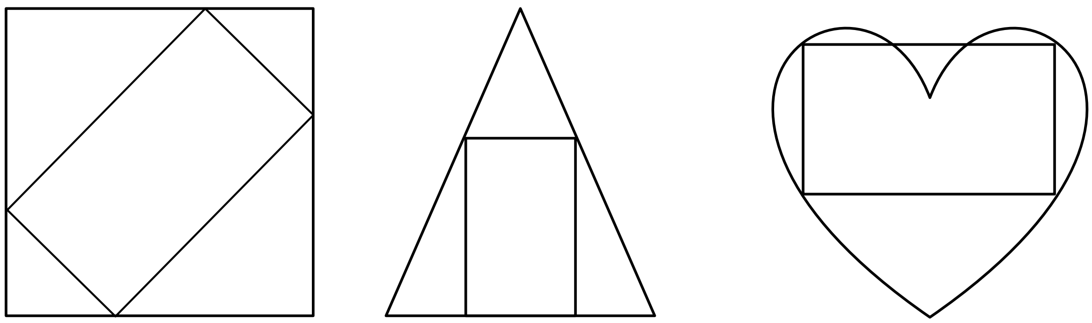
Chapter 3 - Methodology

3.1 Overview

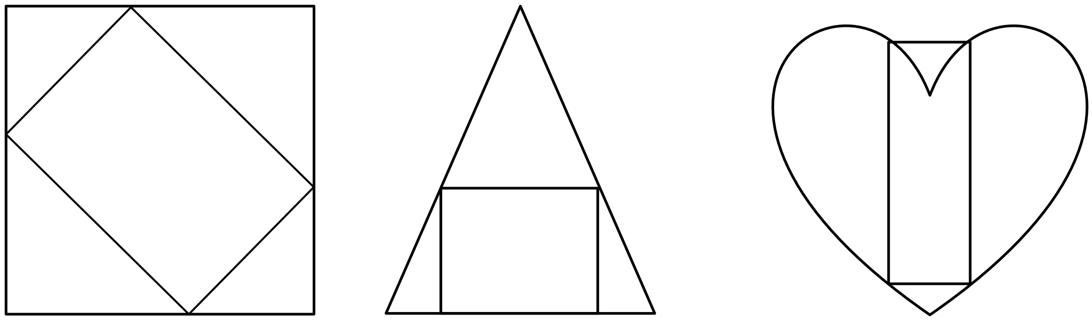
We would give the proof for the inscribed squares in the Jordan curves used in this project and calculate the areas of the squares and Jordan curves to formulate an equation.

3.2 Proof for inscribed squares

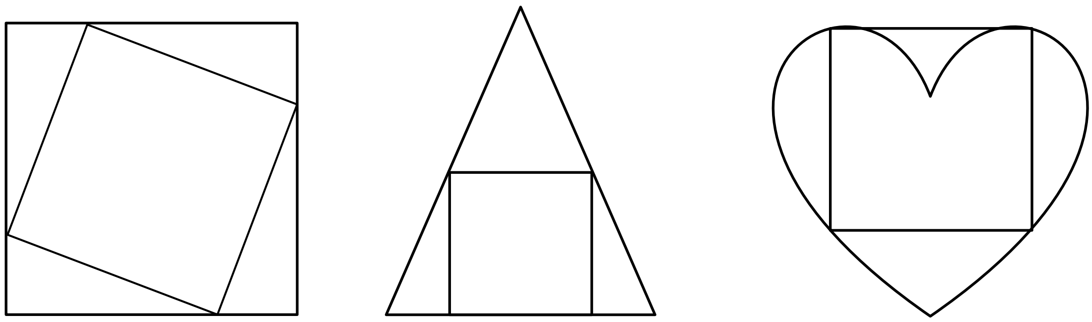
Due to the inscribed rectangle theorem, a rectangle can be inscribed in any Jordan curve.



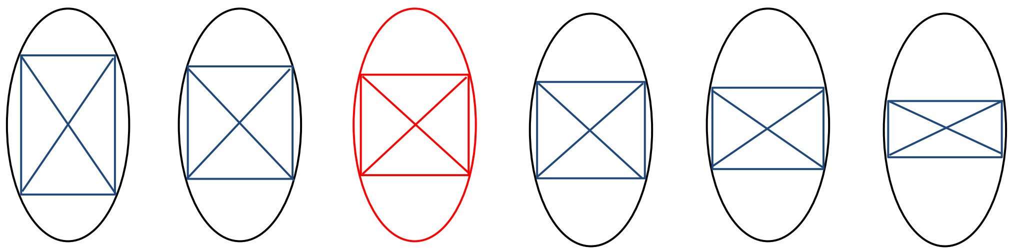
For the Jordan curves (triangle, square, circle, ellipse, heart) we are using and other symmetrical closed curves, another rectangle with the same orientation can be drawn but with the height of the rectangle now longer than the width, or vice versa if the first rectangle was already constructed as such.

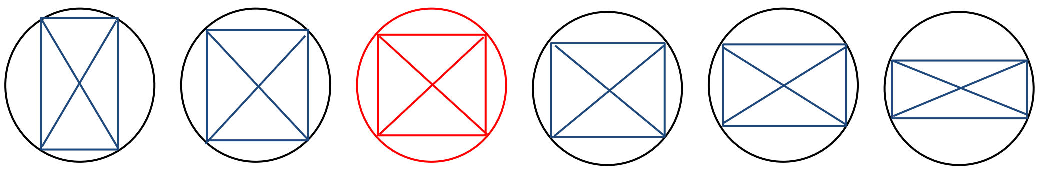


Due to the intermediate value theorem, we are able to conclude that there is an inscribed rectangle in which its height and width is equal, thus producing an inscribed square.

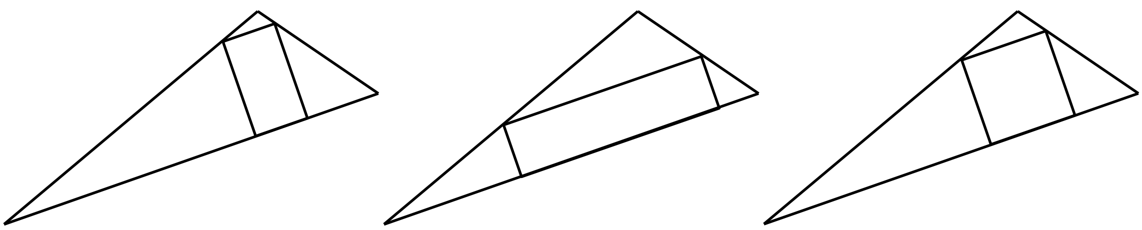


For example, a rectangle can be inscribed inside a circle and ellipse with the height greater than the width if it hugs the major axis. However, if the rectangle is hugging the minor axis the width will be greater than the height. If we keep dragging the diagonals of the rectangle, eventually they will equate to each other, due to the intermediate value theorem.





Even though not all triangles are symmetrical, the concept above can be applied to asymmetrical triangles as well to find an inscribed square.



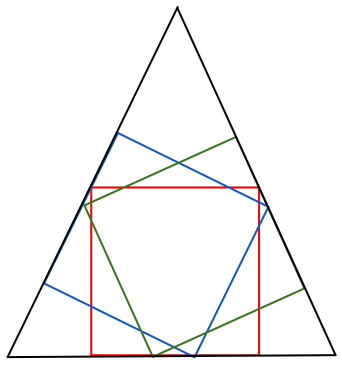
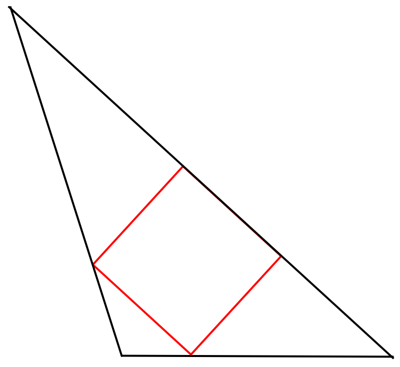
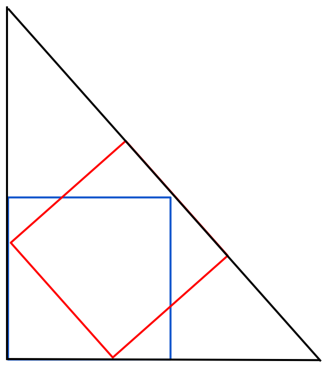
Thus, this proves that the triangle, square, circle, ellipse, heart do have an inscribed square and we will proceed to find the area of the inscribed squares and Jordan curves.

3.3 Area of Inscribed Squares and Jordan Curves

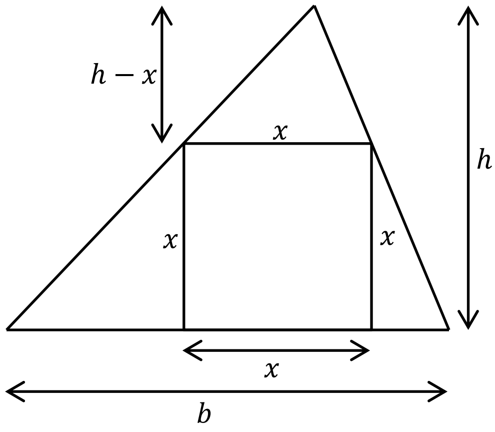
To find the area of the inscribed squares we use the formula of the area of the Jordan curves and device a formula to calculate the area of the inscribed square using the dimensions of the area of the Jordan curves.

3.3.1 Triangle

Acute triangles have 3 inscribed squares, obtuse triangles have 1 and right-angled triangles have 2 inscribed squares.

		
Acute Triangle	Obtuse triangle	Right-angled Triangle

For all inscribed squares whose one side completely touches a side of the triangle



$$Area\ of\ triangle = \frac{bh}{2}$$

$$\frac{bh}{2} = \frac{x(h-x)}{2} + \frac{s(b-x)}{2} + x^2$$

$$bh = x(h-x) + x(b-x) + 2x^2$$

$$= hx - x^2 + bx - x^2 + 2x^2$$

$$= hx + bx$$

$$= x(h+b)$$

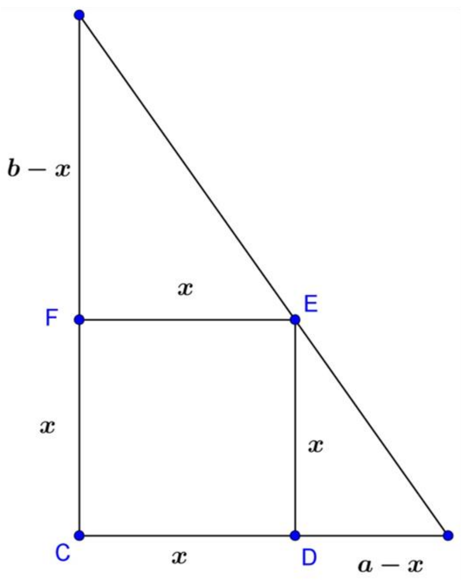
$$x(h+b) = bh$$

$$x = \frac{bh}{b+h}$$

$$Area\ of\ inscribed\ square = x^2$$

$$= \left(\frac{bh}{b+h}\right)^2$$

$$Area\ of\ inscribed\ square\ in\ triangle = \frac{b^2h^2}{b^2+2bh+h^2}$$



For all inscribed squares whose two sides completely touch two sides of the triangle:

By similar triangles,

$$\frac{b}{h} = \frac{x}{h-x}$$

$$hx = bh - bx$$

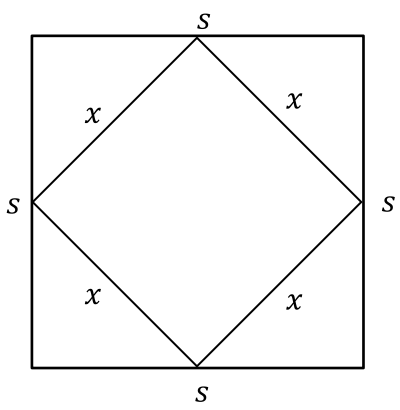
$$x = \frac{bh}{b+h}$$

$$Area\ of\ inscribed\ square = x^2$$

$$Area\ of\ inscribed\ square\ in\ triangle = \frac{b^2h^2}{b^2 + 2bh + h^2}$$

3.3.2 Square

For this part we used the inscribed square with the smallest area



$$Area\ of\ square = s^2$$

$$Using\ Pythagoras\ Theorem$$

$$x^2 = \left(\frac{1}{2}s\right)^2 + \left(\frac{1}{2}s\right)^2$$

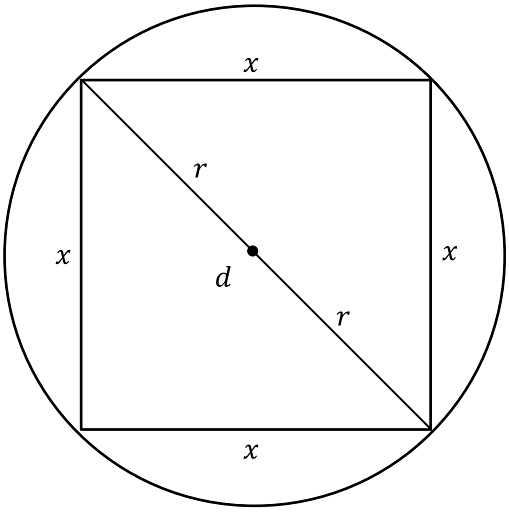
$$x^2 = \frac{1}{4}s^2 + \frac{1}{4}s^2$$

$$x^2 = \frac{1}{2}s^2$$

$$Area\ of\ smallest\ inscribed\ square\ in\ square = \frac{1}{2}s^2$$

3.3.3 Circle

As the vertices of the polygon all touch the circle, the diagonals of the inscribed square are equal to the diameter of the circle. The angle inscribed in a semicircle is always 90°, thus all the angles of the polygon are 90°.



$$Using\ Pythagoras\ Theorem,$$

$$d^2 = x^2 + x^2$$

$$d^2 = 2x^2$$

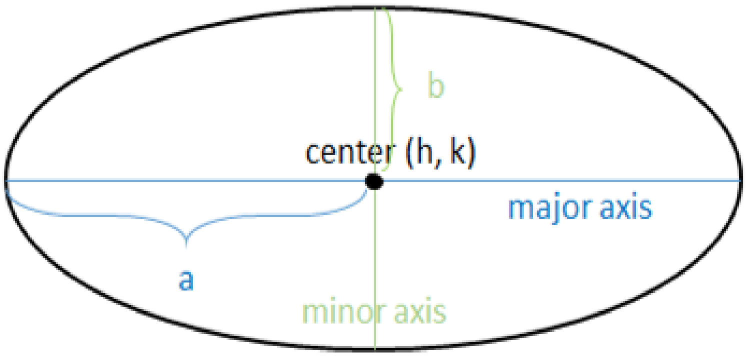
$$x^2 = \frac{d^2}{2}$$

$$x^2 = \frac{(2r)^2}{2}$$

$$x^2 = 2r^2$$

$$Area\ of\ inscribed\ square\ in\ circle = 2r^2$$

3.3.4 Ellipse



$$Equation\ of\ ellipse: \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$$(h, k)\ is\ the\ centre\ point\ of\ the\ ellipse$$

$$When\ (h, k)\ =\ (0,0)$$

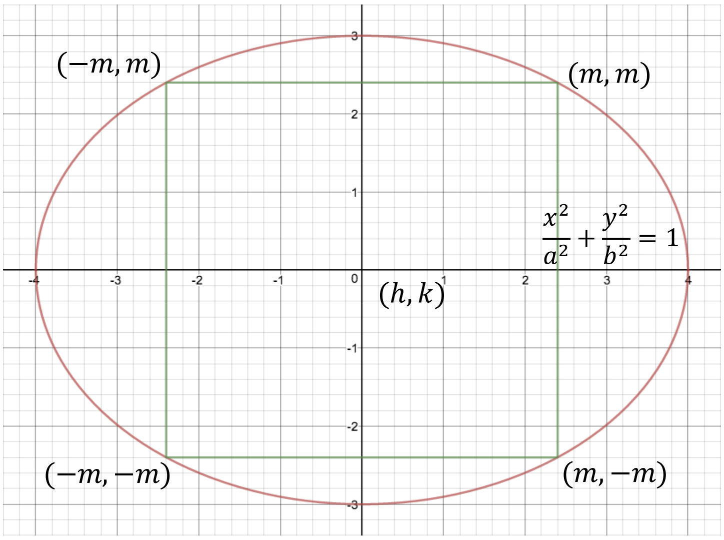
$$Equation\ of\ ellipse: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$Equations\ of\ diagonals\ of\ inscribed\ square\ in\ ellipse: y = x\ , y = -x$$

Let the coordinates of one vertex of the square be (m, n).

Since a square has sides of equal length and the centre of the square is at the origin (0,0),

Hence the coordinates of the vertices of the square (clockwise) are(m, m), $(m, -m)$, $(-m, -m)$ and $(-m, m)$.



$$\therefore x = m - (-m) = 2m$$

$$Diagonal\ of\ square : y = x \_\_\_\_\_(1)$$

$$ellipse : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \_\_\_\_\_(2)$$

$$Sub\ (1)\ into\ (2)$$

$$\frac{x^2}{a^2} + \frac{x^2}{b^2} = 1$$

$$\frac{b^2x^2 + a^2x^2}{a^2b^2} = 1$$

$$\frac{(a^2 + b^2)x^2}{a^2b^2} = 1$$

$$x^2 = \frac{a^2b^2}{a^2 + b^2}$$

$$When\ x = m$$

$$m^2 = \frac{a^2b^2}{a^2 + b^2}$$

$$Area\ of\ inscribed\ square\ in\ ellipse = side^2$$

$$= (2m)^2$$

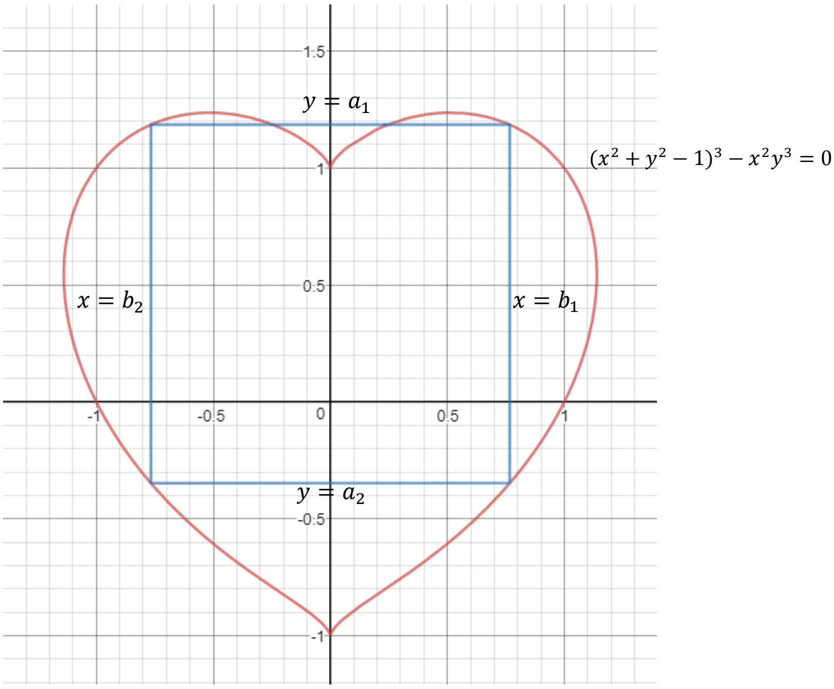
$$= 4m^2$$

$$= \frac{4a^2b^2}{a^2 + b^2}$$

3.3.5 Heart

The heart we are using in this project has the equation $(x^2 + y^2 - 1)^3 - x^2y^3 = 0$

The inscribed square has sides that are parallel to the x and y axes of the Cartesian coordinate system.



$$Equation\ of\ sides\ of\ square: y = a_1$$

$$y = a_2$$

$$x = b_1$$

$$x = b_2$$

$$b_2 = -b_1$$

$$|a_1 - a_2| = |b_1 - (-b_1)|$$

$$|a_1 - a_2| = |2b_1|$$

$$Since\ b_1\ is\ positive,$$

$$|a_1 - a_2| = 2b_1 \_\_\_\_\_\_(1)$$

$$Equation\ of\ heart : (x^2 + y^2 - 1)^3 - x^2y^3 = 0 \_\_\_\_\_\_(2)$$

$$Sub\ x = b_1\ into\ (2)$$

$$\left({b_1}^2 + y^2 - 1\right)^3 - {b_1}^2y^3 = 0$$

$$\left({b_1}^2 + y^2 - 1\right)^3 = {b_1}^2y^3$$

$${b_1}^2 + y^2 - 1 = b_1^{\frac{2}{3}}y$$

$$y^2 - b_1^{\frac{2}{3}}y + {b_1}^2 - 1 = 0$$

$$y = \frac{-\left(-b_1^{\frac{2}{3}}\right) \pm \sqrt{\left(b_1^{\frac{2}{3}}\right)^2 - 4\left({b_1}^2 - 1\right)}}{2}$$

$$y = \frac{b_1^{\frac{2}{3}} \pm \sqrt{b_1^{\frac{4}{3}} - 4{b_1}^2 + 4}}{2}$$

$$a_1 = \frac{b_1^{\frac{2}{3}} + \sqrt{b_1^{\frac{4}{3}} - 4{b_1}^2 + 4}}{2}\ ,\ a_2 = \frac{b_1^{\frac{2}{3}} - \sqrt{b_1^{\frac{4}{3}} - 4{b_1}^2 + 4}}{2}$$

$$Sub\ a_1 = \frac{b_1^{\frac{2}{3}} + \sqrt{b_1^{\frac{4}{3}} - 4{b_1}^2 + 4}}{2}\ ,\ a_2 = \frac{b_1^{\frac{2}{3}} - \sqrt{b_1^{\frac{4}{3}} - 4{b_1}^2 + 4}}{2}\ \ into\ (1)$$

$$\left|\frac{b_1^{\frac{2}{3}} + \sqrt{b_1^{\frac{4}{3}} - 4{b_1}^2 + 4}}{2} - \frac{b_1^{\frac{2}{3}} - \sqrt{b_1^{\frac{4}{3}} - 4{b_1}^2 + 4}}{2}\right| = 2b_1$$

$$b_1 \approx 0.766618$$

$$b_1 = 0.767\ (3s.f)$$

We can find the area of the square using the value of b_1 but we will continue to calculate a_1 and a_2 so that we know the position of the inscribed square in the heart in the Cartesian coordinate system (finding the coordinates).

$$a_1 \approx \frac{0.766618^{\frac{2}{3}} + \sqrt{0.766618^{\frac{4}{3}} - 4(0.766618)^2 + 4}}{2}$$

$$a_1 \approx 1.18543$$

$$a_1 = 1.19\ (3s.f)$$

$$a_2 = a_1 - 2b_1$$

$$= 1.18543 - 2(0.766618)$$

$$\approx -0.347806$$

$$a_2 = -0.348\ (3s.f)$$

$$\therefore Coordinates\ of\ vertices\ of\ inscribed\ square:$$

$$(0.767\ , 1.19), (0.767\ , -0.348), (-0.767\ , -0.348), (-0.767\ , 1.19)$$

$$\therefore Area\ of\ inscrcibed\ square\ in\ heart =\ side^2$$

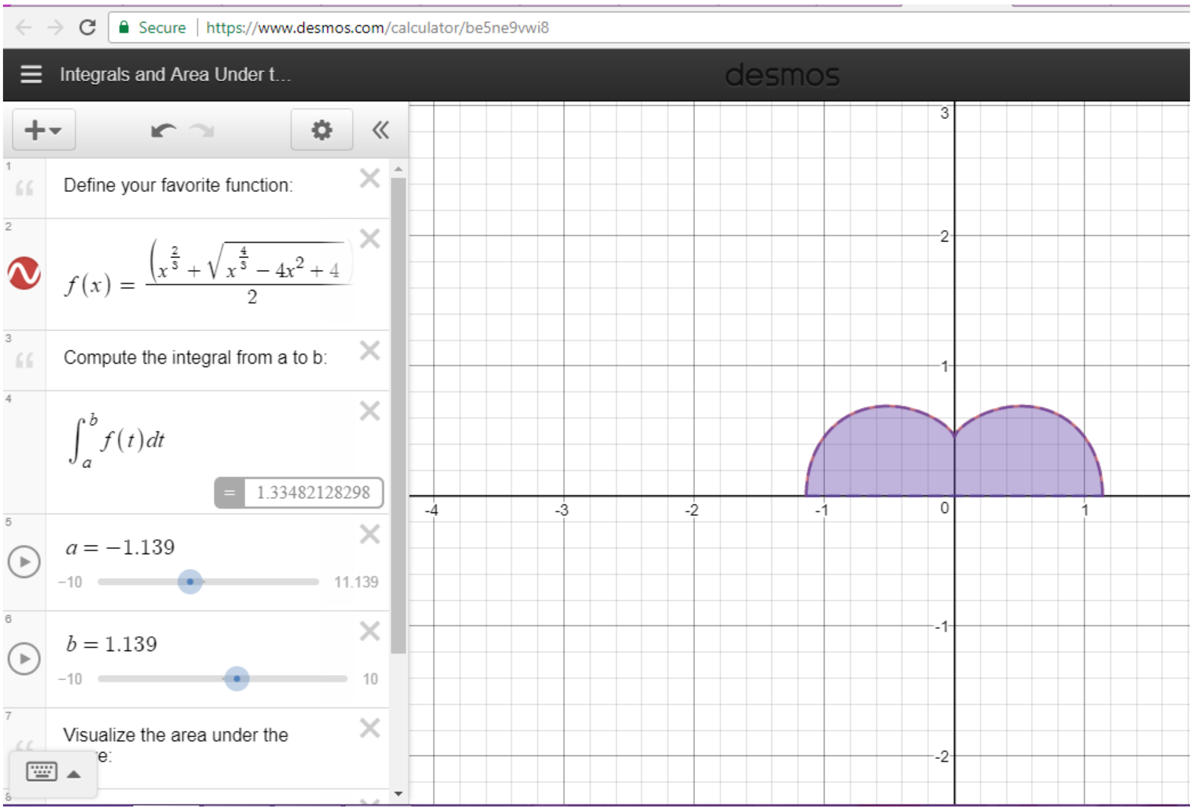
$$= (2b_1)^2$$

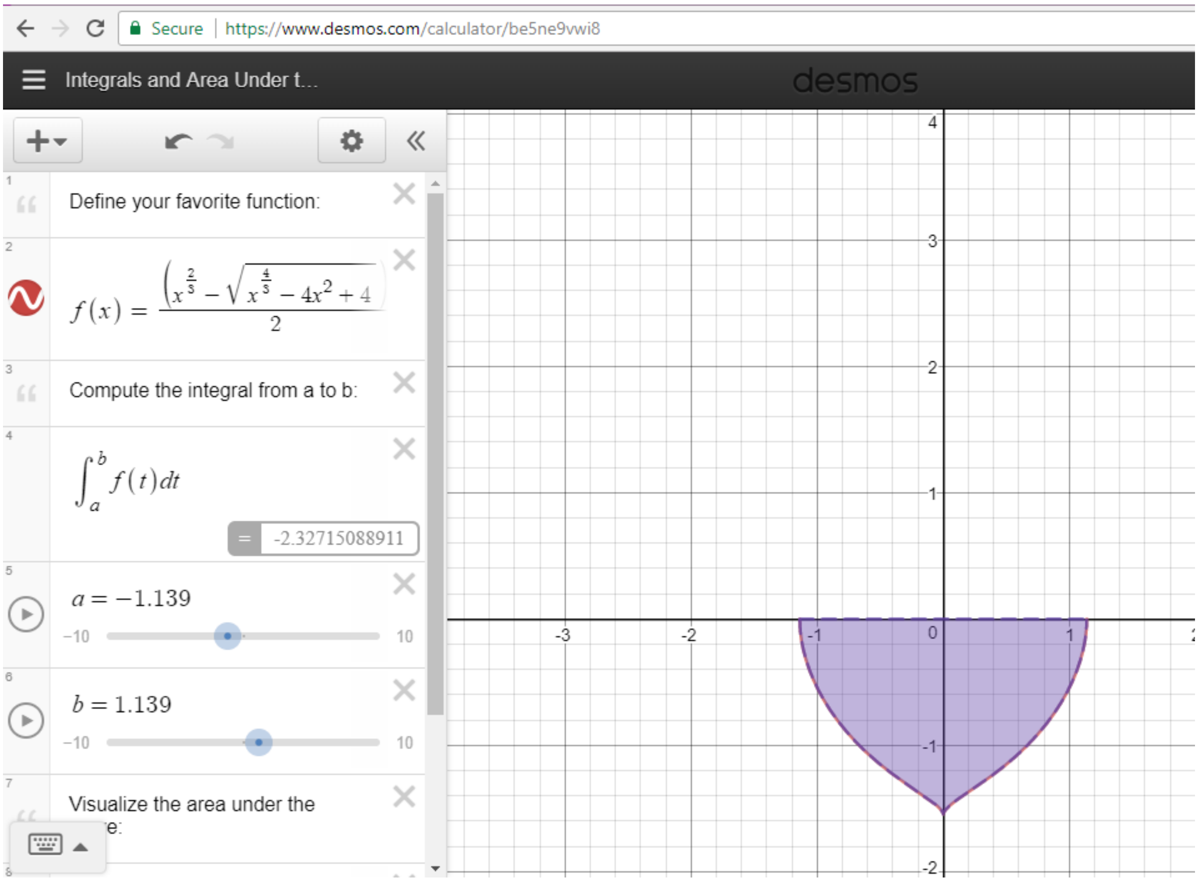
$$= [2(0.766618)]^2$$

$$\approx 2.35081$$

$$= 2.35\ units^2\ (3s.f)$$

To find the area of the heart, we used Desmos which computes the area using integration with the formula: $Area = \int_a^b f(x)\,dx$





$$Area\ of\ heart = Area\ of\ top\ part + Area\ of\ bottom\ part$$

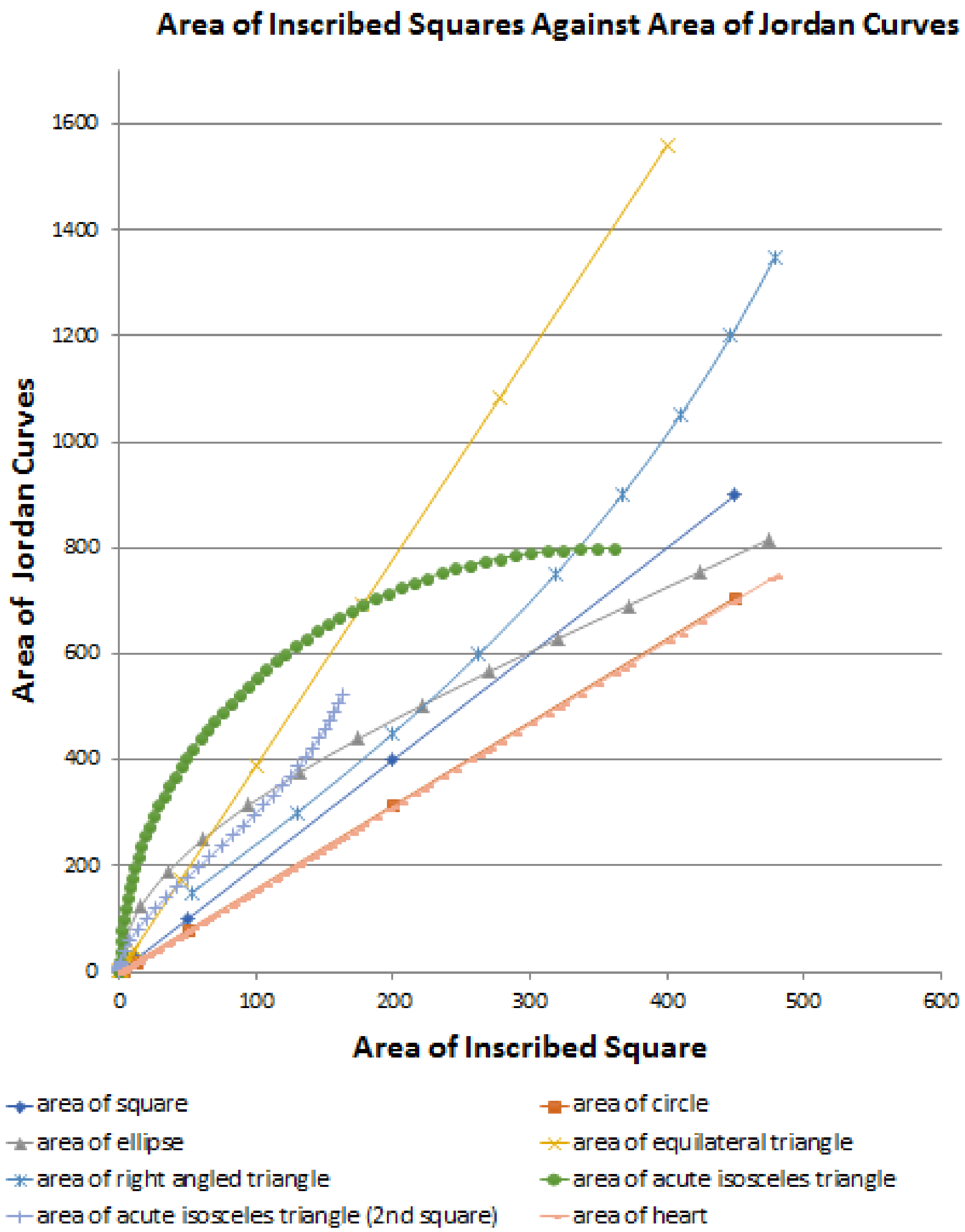
$$\approx 1.33482 + 2.32715$$

$$\approx 3.66197$$

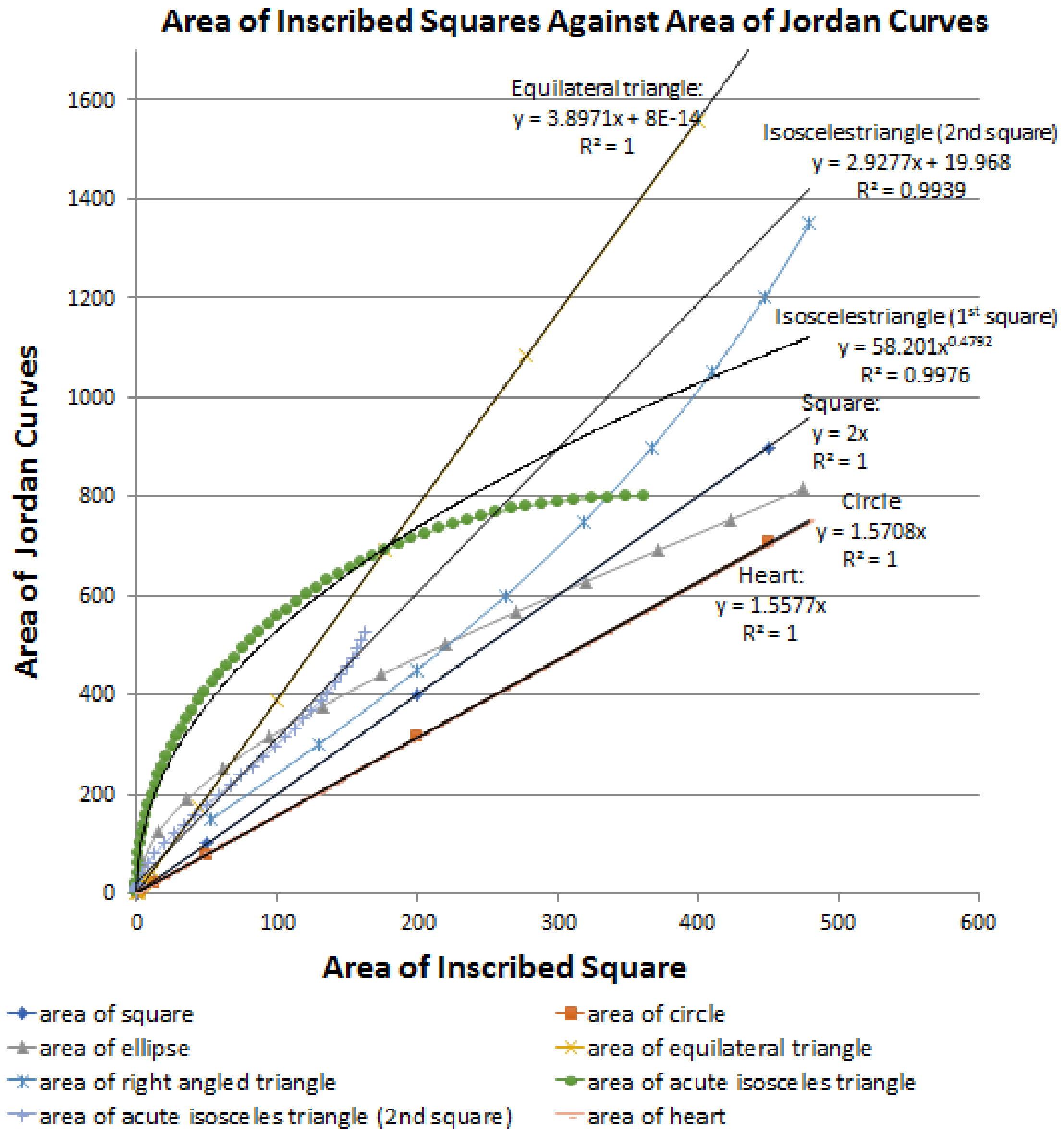
$$= 3.66\, units^2\ (3s.f)$$

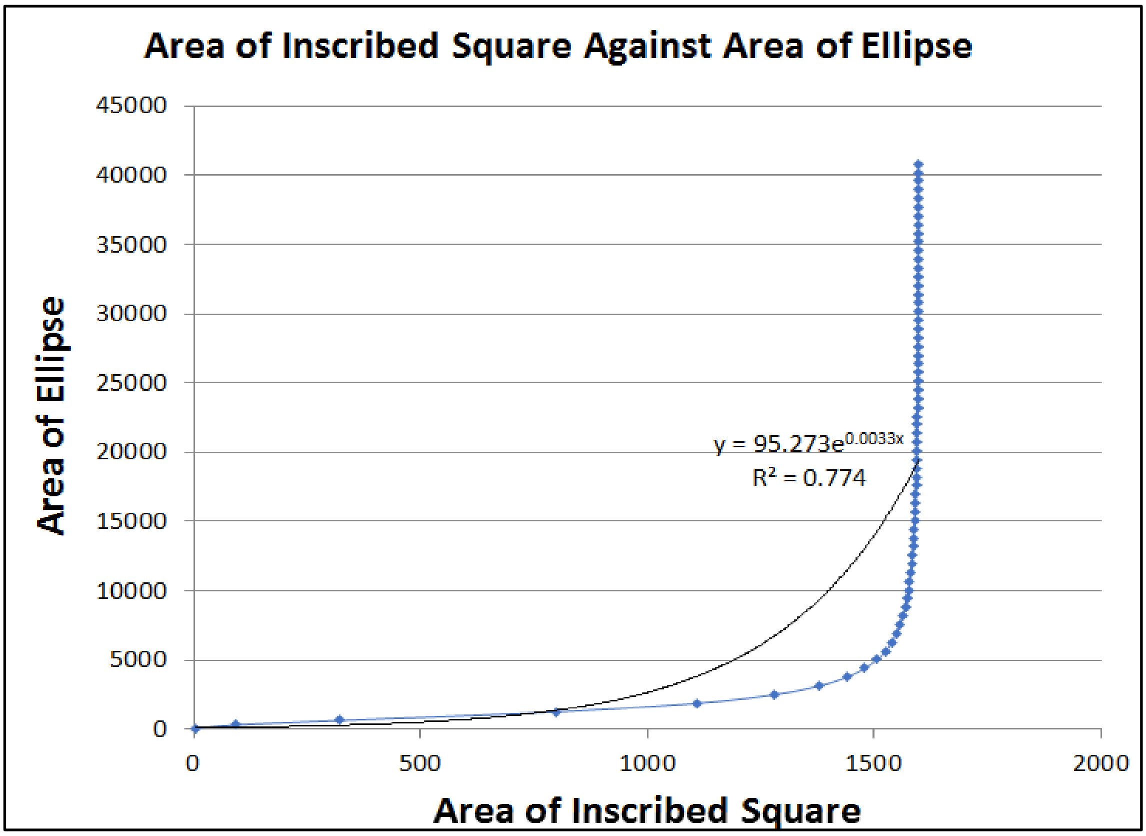
3.4 Graphs of Area of Inscribed Squares and Jordan Curves

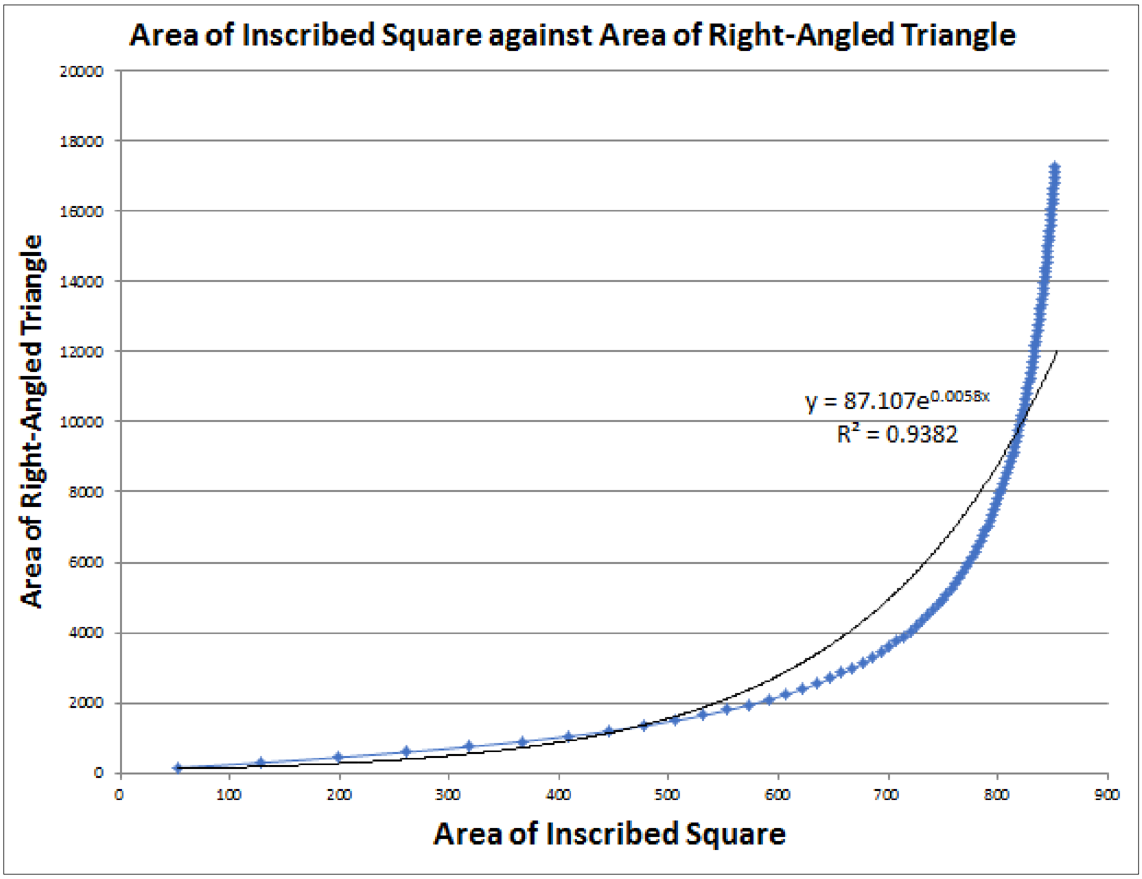
We graphed the areas of the inscribed squares against the areas of the Jordan curves. For the triangles and ellipses, we used different values for their dimensions (hypotenuse and height of triangle and minor axis of ellipse) to produce different shapes, areas of the Jordan curves and areas of their inscribed squares. For the triangles we used base lengths of 30 units, while we used 20 units as the major axes of the ellipses.



Using Microsoft Excel, we found the best fit lines and curves for the graphs and their equations, shown in the graphs and table below. Other graphs for the ellipse and right-angled triangle was made to show a better curve for the overall trend of the curves.




Area of Inscribed Square Against Area of Ellipse
45000
40000
35000
30000
25000
20000
15000
10000
5000
0
Area of Ellipse
y = 95.273e^0.0033x
R² = 0.774
0
500
1000
1500
2000
Area of Inscribed Square


Area of Inscribed Square against Area of Right-Angled Triangle
20000
18000
16000
14000
12000
10000
8000
6000
4000
2000
0
Area of Right-Angled Triangle
y = 87.107e^0.0058x
R² = 0.9382
0
100
200
300
400
500
600
700
800
900
Area of Inscribed Square

Jordan Curve	**Equation**
Equilateral triangle	$y = 3.8971x + 8\times10^{-14}$
Right-angled triangle	$y = 87.107e^{0.0058x}$
Isosceles triangle (1st square)	$y = 58.201x^{0.4792}$
Isosceles triangle (2nd square)	$y = 19.148e^{0.02x}$
Square	$y = 2x$
Circle	$y = 1.5708x$
Ellipse	$y = 95.273e^{0.0033x}$
Heart	$y = 1.5577x$

3.5 Linear Graphs of the Equations

We have found the equations for the above Jordan curves but we may proceed to find the linear graphs for those equations to conclude all the equations into one formula.

For the exponential functions, we manipulated the equations using log transformation before we change them into linear graphs.

From exponential functions in the format of $y = a\, e^{bx}$, we add log to both sides

$$y = a\, e^{bx}$$

$$\log y = \log a\, e^{bx}$$

$$\log y = \log a\ + \log\ e^{bx}$$

$$\log y = \log a\ + bx\ \log\ e$$

$$\log y = \log a\ + bx$$

$$\log y = bx + \log a$$

$$\therefore Y = mX + c, \text{ where } Y = \log y\,, m = b, X = x \text{ and } c = \log a$$

Substituting the exponential function from the table into the format above, we can get linear graphs.

For the power function, in the format of $y = a\, x^b$, we also use log transformation and add log to both sides of the equation.

$$y = a\, x^b$$

$$\log y = \log a\, x^b$$

$$\log y = \log a\ + \log x^b$$

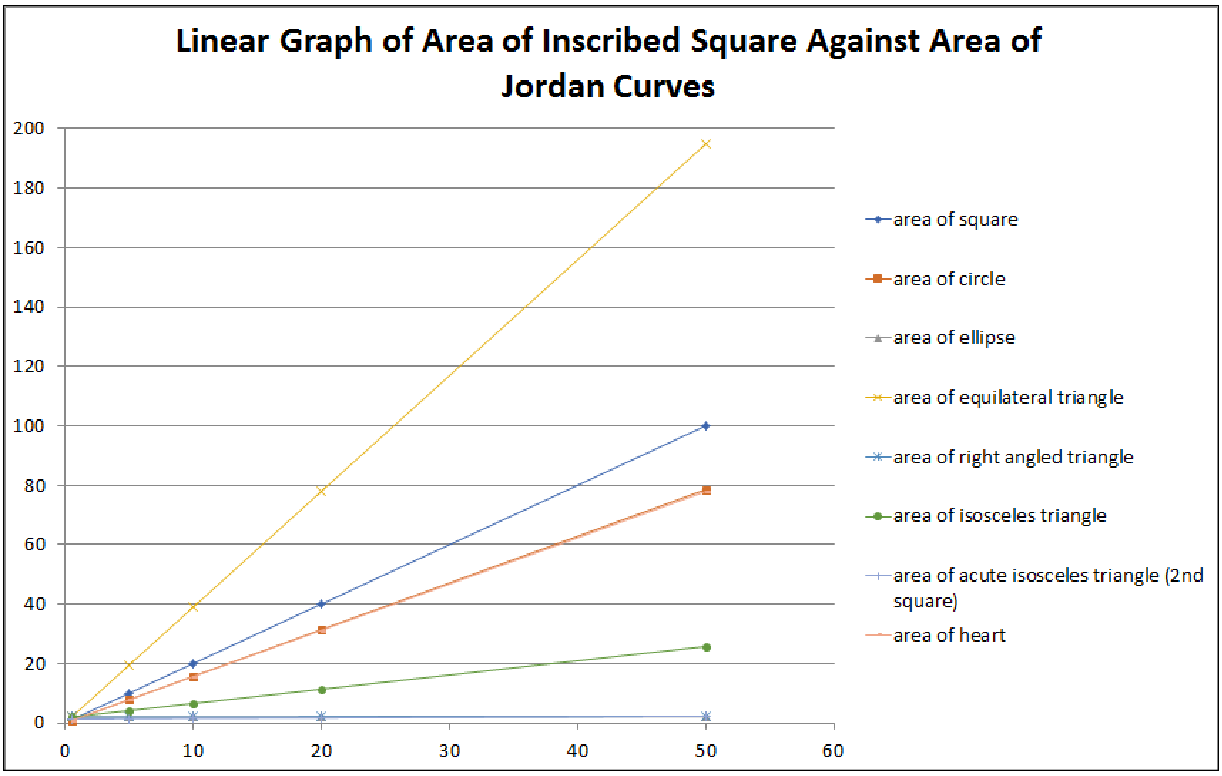
$$\log y = \log a\ + b \log x$$

$$\log y = b \log x + \log a$$

$$\therefore Y = mX + c, \text{ where } Y = \log y\,, m = b, X = \log x \text{ and } c = \log a$$

Substituting the power function from the table into the format above, we can then produce the table and graph of linear functions for the areas of the Jordan curves and inscribed squares.

Jordan Curve	Equation	Linear Equation / Equation in Log form
Equilateral triangle	$y = 3.8971x$	$y = 3.8971x$
Right-angled triangle	$y = 87.107e^{0.0058x}$	$\log y = 0.0058x + \log 87.107$ $Y = 0.0058X + \log 87.107$
Isosceles triangle (1st square)	$y = 58.201x^{0.4792}$	$\log y = 0.4792 \log x + \log 58.201$ $Y = 0.4792X + \log 58.201$
Isosceles triangle (2nd square)	$y = 2.9277x + 19.968$	$y = 2.9277x + 19.968$
Square	$y = 2x$	$y = 2x$
Circle	$y = 1.5708x$	$y = 1.5708x$
Ellipse	$y = 95.273e^{0.0033x}$	$\log y = 0.0033x + \log 95.273$ $Y = 0.0033X + \log 95.273$
Heart	$y = 1.5577x$	$y = 1.5577x$



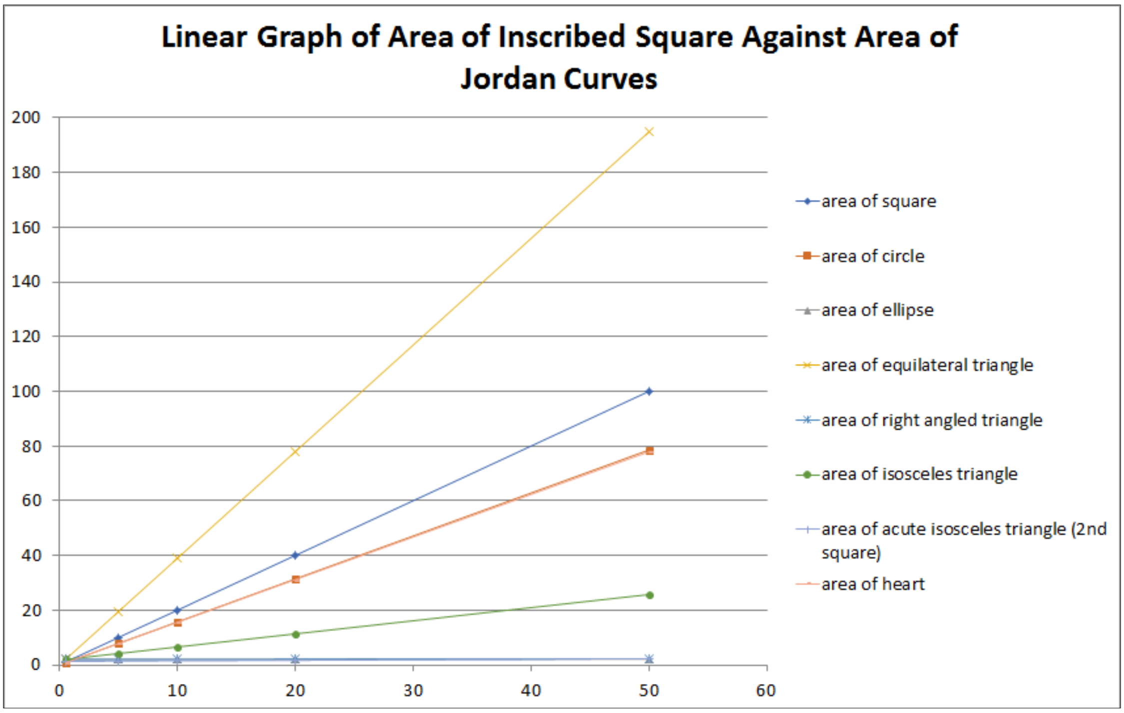
Chapter 4 – Results and Analysis

4.1 Analysis of Results

We have found the formulas for the relationship between the area of some Jordan curves and their inscribed squares as shown in the table below.

Jordan Curve	Equation
Equilateral triangle	$y = 3.8971x + 8\times10^{-14}$
Right-angled triangle	$y = 87.107e^{0.0058x}$
Isosceles triangle (1st square)	$y = 58.201x^{0.4792}$
Isosceles triangle (2nd square)	$y = 19.148e^{0.02x}$
Square	$y = 2x$
Circle	$y = 1.5708x$
Ellipse	$y = 95.273e^{0.0033x}$
Heart	$y = 1.5577x$

We tried finding a pattern to conclude them all into one equation using log transformation and linear graphs to produce all the equations into the format $Y = mX + c$, yet we could not find one definite equation to specify the relationship between the areas of the Jordan curves and inscribed squares as the equations had different functions.



4.2 Limitations and Recommendations

The equations for the ellipse and isosceles triangle were only produced from using one value for one of their dimensions (base length and major axis, respectively) and thus would not be able to calculate the areas of their inscribed squares if the value of those dimensions were changed.

We would suggest finding equations for those two figures that would accommodate for any values of their dimensions, as well as an equation that would be able to combine the equations for the areas of all the Jordan curves in this project and their inscribed squares.

Chapter 5 - Conclusion

Through the use of the Inscribed Rectangle Theorem and the Intermediate Value Theorem, we were able to prove that there are inscribed rectangle in the Jordan curves triangle, square, circle, ellipse and heart. We also came up with equations to calculate the areas of their inscribed squares from the dimensions of the Jordan curves as in the table in page 21, by exploring the equations of the areas of the Jordan curves, plotting them into graphs and estimating their equations from their trend lines or best fit curves.

However, due to some constraints, we could not find a suitable formula to put the equations into one statement when we used log transformation to further make the equations into linear graphs and into the form of $Y = mX + c$

To continue this project, we suggest that the equations could be put together into one formula that would be able to summarise all of them.

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