Napoleon's Theorem 'Addict'

Submitted by S3 PLMGS (S) Students

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Abstract

In mathematics, Napoleon's theorem states that if equilateral triangles are constructed on the sides of any triangle, either all outward, or all inward, the centers of those equilateral triangles themselves form an equilateral triangle.

In this project which we carried out last year, we came up with an idea of applying Napoleon's Theorem to polygons other than three-sided figures. We constructed equilateral triangles outwardly on the sides of regular polygons (which we called the "Original Polygon"), and connected the centers of the equilateral triangles to form another regular polygon (which we called the "Constructed Polygon"). A formula connecting the lengths of the sides of the Original Polygon and the sides of Constructed Polygon was found. Another formula was found when we changed the equilateral triangles to regular polygons which had the same number of sides as the Original Polygon (e.g. if the Original Polygon was a pentagon, regular pentagons would be constructed outwardly at each side of the original pentagon). However, due to the limitation of our knowledge and constrain of time, we were unable to prove these formulae vigorously and explore Napoleon's Theorem any further.

Aiming to further extend on our findings, we constructed equilateral triangles inwardly on the sides of regular polygons this year in addition to outward constructions. Besides looking into the relationship of the lengths of the Original Polygons and the Constructed Polygons, the connection between their areas was also taken into consideration.

Acknowledgement

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Chapter 1 Introduction

1.1 Objective

Based on each side of a regular polygon we constructed equilateral triangles either outward or inward, then connect all of the centers of the equilateral triangles to get a new polygon, and we try to find the relationship of length ratio and area ratio between the new polygon and the original polygon according to Napoleon's theorem.

1.2 Problem

Our problem is the Napoleon's theorem only stop at triangles, but we try to apply it in regular polygonal.

1.3 Background

Napoleon is best known as a military genius and Emperor of France but he was also an outstanding mathematics student. He was born on the island of Corsica and died in exile on the island of Saint-Hélène after being defeated in Waterloo. He attended school at Brienne in France where he was the top mathematics student. He took algebra, trigonometry and conics but his favorite was geometry. After graduation from Brienne, he was interviewed by Pierre Simon Laplace(1749-1827) for a position in the Paris Military School and was admitted by virtue of his mathematics ability. He completed the curriculum, which took others two or three years, in a single year and subsequently he was appointed to the mathematics section of the French National Institute.

Chapter 2: Literature Review

2.1 Overview

There are three parts to our Literature Review, namely Napoleon's Theorem, Similar Triangles and Trigonometry.

2.2.1 Napoleon's Theorem

The theorem is often attributed to Napoleon Bonaparte (1769–1821). However, it may just date back to W. Rutherford's 1825 question published in The Ladies' Diary, four years after the French emperor's death.

The following entry written by Mr. W. Rutherford Woodburn appeared on page 47 in The Ladies' Diary: "Describe equilateral triangles (the vertices being either all outward or all inward) upon the three sides of any triangle *ABC*: then the lines which join the centers of gravity of those three equilateral triangles will constitute an equilateral triangle. This is the earliest known reference to Napoleon's theorem. It was believed that Napoleon Bonaparte further extended this theorem and named it after himself.

Napoleon's Theorem states that if equilateral triangles are constructed on the sides of any given triangle, either all outward, or all inward, the centers of those equilateral triangles themselves form an equilateral triangle.

The triangle thus formed is called the Napoleon triangle (inner and outer). In addition, the difference in area of these two triangles equals the area of the original triangle.

2.2.2 How Napoleon's Theorem works

For example, if △ABC is the original triangle, construct equilateral triangles △ABE, △BCF and △ACD outwardly at the sides of △ABC. Point H, point I and point G are the centers of △ABE, △BCF and △ACD respectively.

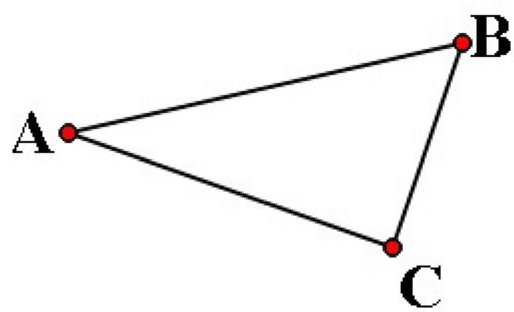

Figure 2.2.2(1) a triangle, △ABC

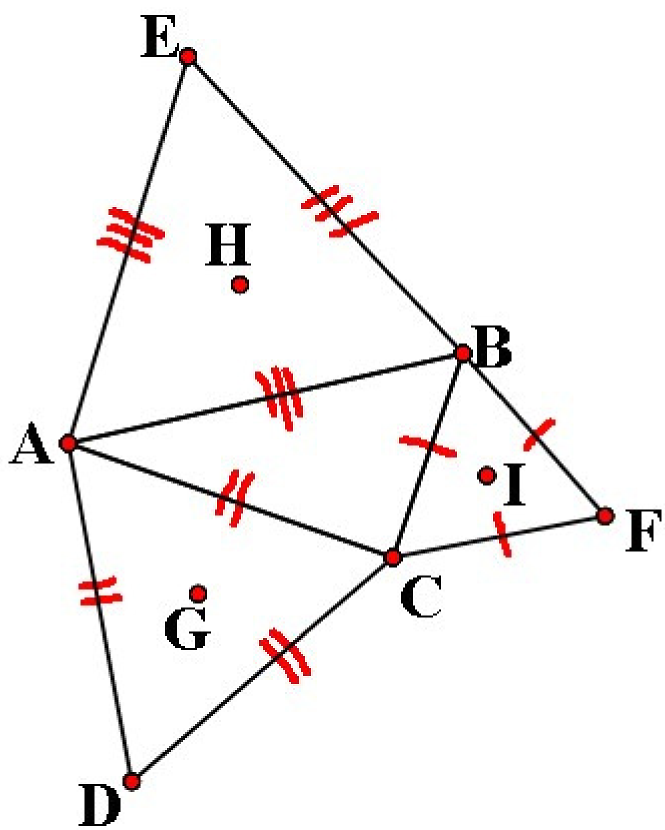

Figure 2.2.2 (2) △ABC with equilateral triangles constructed outwardly at each side

Connecting the centers of these three equilateral triangles, another equilateral triangle △GHI is formed.

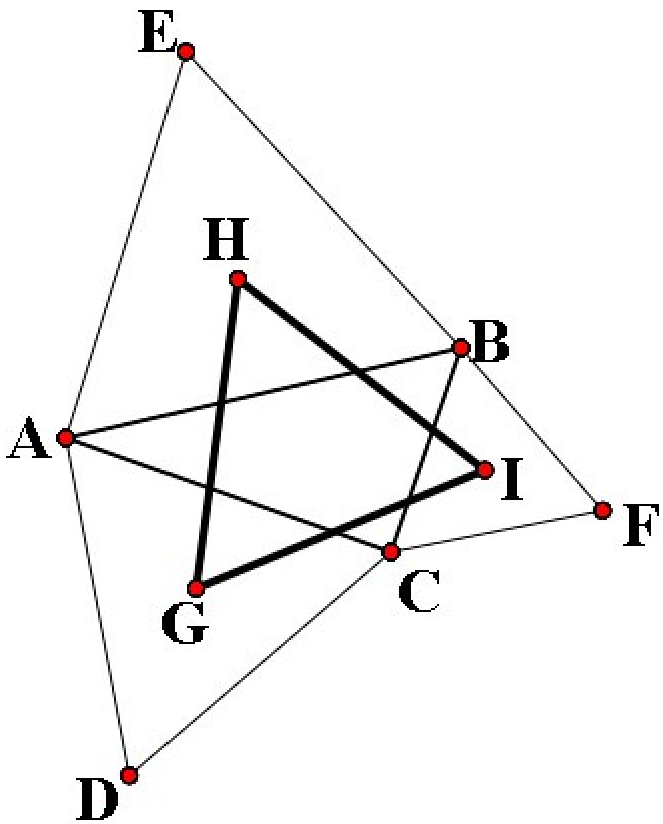

Figure 2.2.2 (3) an outward Napoleon triangle △GHI is formed based on the original △ABC

Secondly, using the same original triangle △ABC, equilateral triangles △ABN, △BCL and △ACM are constructed inwardly at the sides of △ABC. Point Y, point Z and point X are the centers of △ABN, △BCL and △ACM respectively.

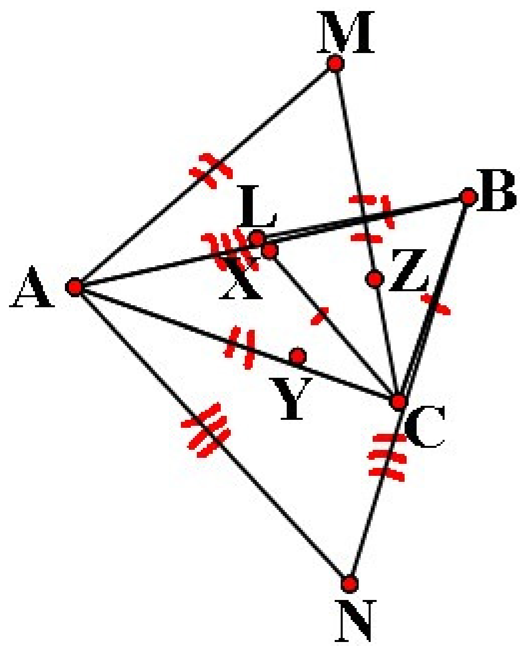

Figure 2.2.2 (2) △ABC with equilateral triangles constructed inwardly at each side

Connecting the centers of these three equilateral triangles, another equilateral triangle △XYZ is formed.

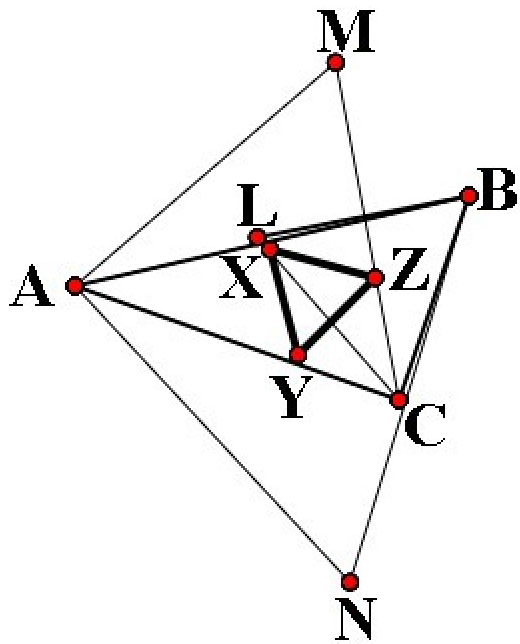

Figure 2.2.2 (5) an inward Napoleon triangle △XYZ is formed based on the original △ABC

According to Napoleon's Theorem, △GHI and △XYZ are called Napoleon triangles. The difference in area of these two Napoleon triangles equals the area of the original triangle, which is △ABC i.e.

Area of △GHI – Area of △XYZ = Area of △ABC.

2.3 Similar Figures

Two triangles $\triangle ABC$ and $\triangle DEF$ are said to be similar if either of the following equivalent conditions holds:

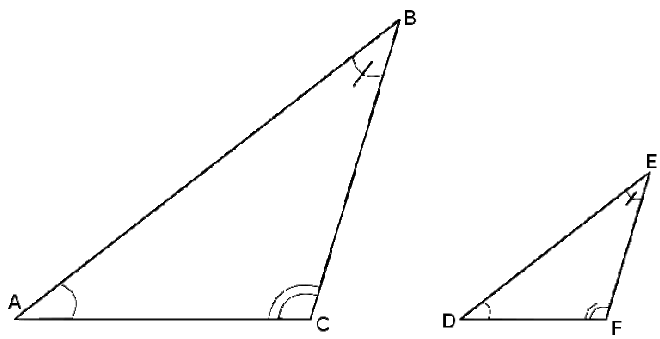

Figure 2.3 Similar triangles

Two pairs of corresponding angles of the two triangles are equal, which implies that the third pair of corresponding angles are also equal. For instance:

$\angle BAC$ is equal in measure to $\angle EDF$, and $\angle ABC$ is equal in measure to $\angle DEF$. This also implies that $\angle ACB$ is equal in measure to $\angle DFE$.

2. Corresponding sides have lengths in the same ratio:

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}.$$

This is equivalent to saying that one triangle (or its mirror image) is an enlargement of the other.

3. Two sides have lengths in the same ratio, and the angles included between these sides have the same measure. For instance:

$$\frac{AB}{DE} = \frac{BC}{EF}$$

and $\angle ABC$ is equal in measure to $\angle DEF$.

The concept of similarity extends to polygons with more than three sides. Given any two similar polygons, corresponding sides taken in the same sequence are proportional and corresponding angles taken in the same sequence are equal in measure. However, proportionality of corresponding sides is not by itself sufficient to prove similarity for polygons beyond triangles (otherwise, for example, all rhombi would be similar). Likewise, equality of all angles in sequence is not sufficient to guarantee similarity (otherwise all rectangles would be similar). A sufficient condition for similarity of polygons is that corresponding sides and diagonals are proportional.

2.4 Trigonometric Functions

2.4.1 Trigonometry

Trigonometry (from Greek trigōnon "triangle" + metron "measure") is a branch of mathematics that studies triangles and the relationships between their sides and the angles between these sides. Trigonometry defines the trigonometric functions, which describe those relationships and have applicability to cyclical phenomena, such as waves. The field evolved during the third century BC as a branch of geometry used extensively for astronomical studies. It is also the foundation of the practical art of surveying.

2.4.2 Sine, Cosine and Tangent

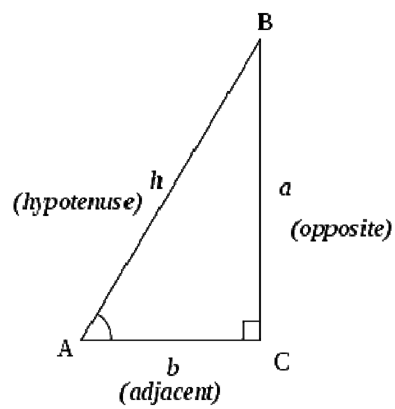

Figure 2.4 a right-angled triangle

The sine of an angle is the ratio of the length of the opposite side to the length of the hypotenuse. (The word comes from the Latin sinus for gulf or bay, since, given a unit circle, it is the side of the triangle on which the angle opens.) In our case

$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{a}{h}.$$

Note that this ratio does not depend on size of the particular right triangle chosen, as long as it contains the angle A, since all such triangles are similar.

The cosine of an angle is the ratio of the length of the adjacent side to the length of the hypotenuse: so called because it is the sine of the complementary or co-angle. In our case

$$\cos A = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{b}{h}.$$

The tangent of an angle is the ratio of the length of the opposite side to the length of the adjacent side: so called because it can be represented as a line segment tangent to the circle that is the line that touches the circle, from Latin linear tangents or touching line (cf. tangere, to touch). In our case

$$\tan A = \frac{\text{opposite}}{\text{adjacent}} = \frac{a}{b}.$$

Chapter 3: Methodology

3.1 Overview

We are going to try to construct equilateral triangles outwardly and inwardly on n-sided polygons. For illustration, we will only use regular octagons in this chapter. Please refer to appendix for the other polygons. After constructing all the inward and outward triangles of *n* -sided polygons, we found the relationship between length of the original polygon and constructed polygons (inward or outward), the relationship between the difference in area of two constructed polygons and the original polygon.

3.2 Construction of outer and inner polygons

When *n* is 8, here is a regular octagon, equilateral triangles are then formed on the sides of the octagon.

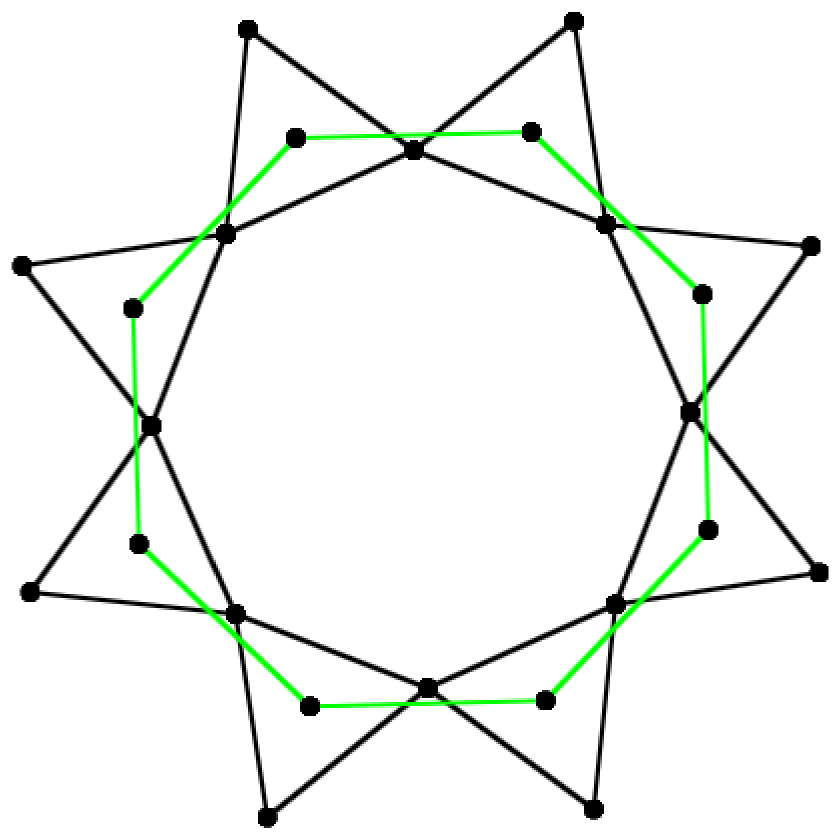

Figure 3.1 Constructing equilateral triangles outward on the sides of a regular polygon

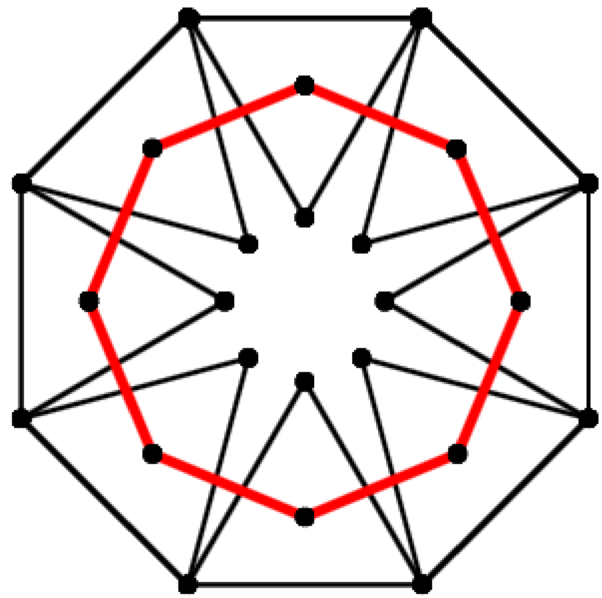

Figure 3.2 Constructing equilateral triangles within a regular octagon

3.3 Length ratio of the constructed outer polygon to the original regular polygon

Consider a regular octagon *ABCDEFGH* being the original regular *n* - sided polygon. Let polygon *IJKLMNOP* be the constructed polygon which is similar as the original octagon (Refer to **Figure 3.1**). We find *Q* which is the centroid of the original regular polygon. Link *QP* and *QI* (*P* and *I* are vertexes of the constructed octagon), *QI* intersects *AB* at *R*. Link *AQ* and *BQ* (i.e. *A* and *B* are the vertexes of the original octagon), extend *QA* to intersect *PI* at *S*.

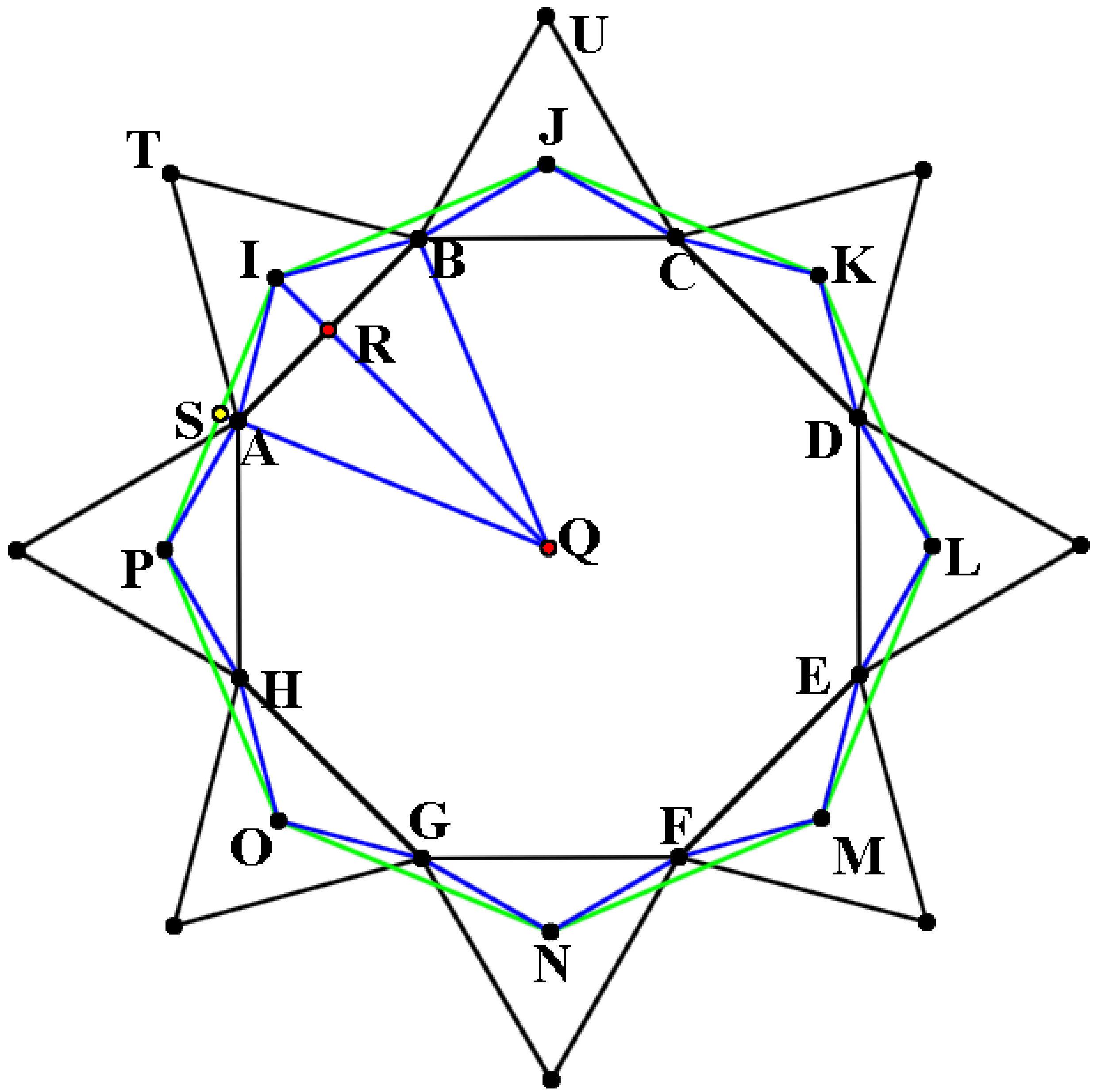

Figure3.3 Constructed octagon

$\measuredangle TAB = \measuredangle TBA = \measuredangle UBC = \measuredangle UCB = 60°$ (equilateral triangles)

I and J are the centroids of the equilateral triangles

Hence, AI 、BI、BJ and CJ bisect $\measuredangle TAB$、$\measuredangle TBA$、$\measuredangle UBC$ and $\measuredangle UCB$ respectively

$\angle IAB = \angle IBA = \angle JBC = \angle JCB = \frac{60°}{2} = 30°$

∵*ABCDRFG* is a regular polygon, we can say that *AB* = *BC.*

In △*AIB* and △*BJC*, ∠*IAB* = ∠*JBC, AB* = *BC*, ∠*IBA* = ∠*JCB,* hence, △*AIB* ≡ △*BJC* (ASA)

Similarly, △*AIB*≡△*BJC* ≡ △*CKD* ≡ △*DLE* ≡ △*EMF* ≡ △*FNG* ≡ △*GOH* ≡ △*HPA,*

hence, *AI* = *IB* = *BJ* = *JC* = *CK* = *KD* = *DL* = *LE* = *EM* = *MF* = *FN* = *NG* = *GO* = *OH* = *HP* = *PA.*

From Figure 3.3, *ABCDEFGH* is a regular polygon, therefore, ∠*ABC* = ∠*BCD*

∠*IBJ*=360° − ∠*IBA* − ∠*JBC* − ∠*ABC* (angles at a point)

And ∠*JCK* = 360°− ∠*JCB* − ∠*KCD* - ∠*BCD*(angles at a point)

∴∠*IBJ*=∠*JCK*

In △*IBJ* and △*JCK, IB* = *JC*, ∠*IBJ* = ∠*JCK, BJ* = *CK,*

therefore, △*IBJ* ≡ △*JCK*（SAS）.

Similarly, △*IBJ* ≡ △*JCK* ≡ △*KDL* ≡ △*LEM* ≡ △*MFN* ≡ △*NGO* ≡ △*OHP* ≡ △*PAI.*

∴ *IJ* = *JK* = *KL* = *LM* = *MN* = *NO* = *OP* = *PI*

∴Octagon *IJKLMNOP* is a regular octagon.

In order to find the relationship between *RQ* and ∠*AIB* & ∠*AQB*, we look at △*AQI* and △*BQI*.

From these two triangles, we see that *AQ* = *BQ, IQ* is the common length and IA = *IB.*

Hence △*AQI* ≡ △*BQI*（SSS）

∴∠*AIQ* = ∠*BIQ,* ∠*AQI* = ∠*BQI*

Therefore we found the relationship that *IQ* is the bisector of both ∠*AIB* and ∠*AQB.*

As both △*ABQ* and △*ABI are* isosceles triangles, *IQ* is perpendicular to *AB.*

Hence ∠*ARQ* = 90°which proved that △*ARQ* is a right-angle triangle.

In order to find the relationship of *SQ* and *IP*

We refer to Figure 3.3, as we proven previously that octagon *IJKLMNOP* is a regular octagon, *PQ* = *IQ.*

Therefore △*PQI* is an isosceles triangle.

In this isosceles triangle, *AQ* bisectors ∠*PQI, h*ence, *QS* is perpendicular to *PI.*

Therefore △*ISQ* is a right-angle triangle.

In the previous working, we found that both △ *ARQ* and △ *IQS* are right-angle triangles.

Here, we want to find the relationship between these two triangles.

In △ *AQR* and △*IQS*, ∠*AQR* is the common angle and ∠*ARQ*=∠*ISQ* = 90°

Therefore, △*AQR* is similar as △*IQS.*

Therefore $\frac{SI}{AR} = \frac{IQ}{AQ}$, and $\frac{SI}{AR}$ is actually the length ratio of constructed polygon and the original one.

In order to find $\frac{SI}{AR}$, we need to find the length of *IQ* and *AQ.*

Let length of IR be *x* units, and the number of sides of the polygon be *n.*

As *IQ* = *RQ* + *IR, the* length of *RQ* is required.

Sin $\angle AIR = \frac{AR}{IA}$

AR = sin 60° $\bullet 2x$

Therefore, *AR* = $\sqrt{3}x$ **(3.1)**

$\angle AQB = \frac{360°}{n}$

$\therefore \angle AQR = \frac{1}{2} \angle AQB = \frac{180°}{n}$ **(3.2)**

In order to find the value of $\frac{IQ}{AQ}$, we need the length of *AQ.*

Sin $\angle AQR = \frac{AR}{AQ}$

Substitute equations **3.1** and **3.3** into this equation.

$$AQ = \frac{\sqrt{3}x}{\sin\frac{180°}{n}} \quad \textbf{(3.3)}$$

We need to find the length of IQ. As *IQ* = *RQ* + *IR* and we let *IR* be *x, the* length of *RQ* is required.

$\because \tan \angle AQR = \frac{AR}{RQ}$

Substitute equations **3.1** and **3.2** into this equation.

$$RQ = \frac{\sqrt{3}x}{\tan\frac{180°}{n}} \quad \textbf{(3.4)}$$

In order to find the value of $\frac{IQ}{AQ}$, and find the length ratio between the constructed polygon and the original polygon, we need the length of *IQ* as we have obtained the length of *AQ* (refer to equation **3.3**)

In Figure 3.3, *QI* = *QR* + *RI*

By substituting equation **3.4** into *QI* = *QR* + *RI.*

$$QI = x + \frac{\sqrt{3}x}{\tan\frac{180°}{n}} \quad \textbf{(3.5)}$$

As we got the value of *IQ* and *AQ* (refer to equations **3.3** and **3.5**), we can hence find the value of $\frac{SI}{AR}$.

$$\frac{SI}{AR}=\frac{IQ}{AQ}$$

$$= \frac{x+\dfrac{\sqrt{3}X}{\tan\frac{180°}{n}}}{\dfrac{\sqrt{3}x}{\sin\frac{180°}{n}}}$$

$$= \frac{1+\dfrac{\sqrt{3}}{\tan\frac{180°}{n}}}{\dfrac{\sqrt{3}}{\sin\frac{180°}{n}}}$$

$$= (1+\frac{\sqrt{3}}{\tan\frac{180°}{n}})\bullet\frac{\sqrt{3}\sin\frac{180°}{n}}{3}$$

$$= \frac{\sqrt{3}\sin\frac{180°}{n}}{3}+\frac{\sin\frac{180°}{n}}{\tan\frac{180°}{n}}$$

$$= \frac{\sqrt{3}}{3}\sin\frac{180°}{n}+\cos\frac{180°}{n}$$

Therefore, length ratio at sides of the constructed outer polygon to the original regular polygon

$=\frac{\sqrt{3}}{3}\sin\frac{180°}{n}+\cos\frac{180°}{n}$.

3.4 Length ratio at sides of the constructed inner polygon to the original regular polygon

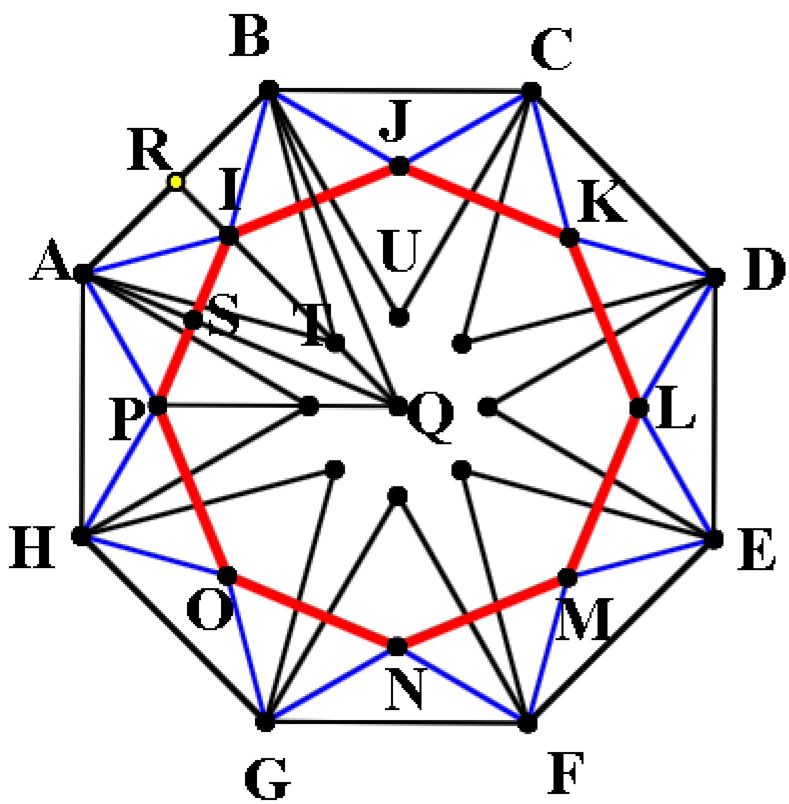

Figure3.4

After constructing equilateral triangles outwardly (refer to **Figure 3.3**), we need to construct equilateral triangles inwardly on an octagon. Let octagon *ABCDEFGH* is the original regular octagon. Polygon *IJKLMNOP* is the constructed polygon by constructing equilateral triangles outwardly on the sides of the original polygon (Refer to **Figure 3.2**). Let *Q* be the centroid of the original regular polygon. Link *QI* and *QP* (*I* and *P* are the vertexes of the constructed octagon), extend *QI* to intersect *AB* at *R*. We link *AQ*、*BQ*(*A* and *B* are the vertexes of the original octagon),and *AQ* intersects *PI* at *S.*

$\angle TAB = \angle TBA = \angle UBC = \angle UCB = 60°$ (equilateral triangles)

I and *J* are the centroids of the equilateral triangles

Hence *AI* 、*BI*、*BJ* and *CJ* bisect $\angle TAB$、$\angle TBA$、$\angle UBC$ and $\angle UCB$ respectively.

$$\angle IAB = \angle IBA = \angle JBC = \angle JCB = \frac{60°}{2} = 30°$$

∵*ABCDRFG* is a regular polygon

Hence, we can say that $AB = BC$

In $\triangle AIB$ and $\triangle BJC$, $\angle IAB = \angle JBC$, $AB = BC$, $\angle IBA = \angle JCB$

Hence, $\triangle AIB \equiv \triangle BJC$ (ASA)

Similarly, $\triangle AIB \equiv \triangle BJC \equiv \triangle CKD \equiv \triangle DLE \equiv \triangle EMF \equiv \triangle FNG \equiv \triangle GOH \equiv \triangle HPA$

Hence, $AI = IB = BJ = JC = CK = KD = DL = LE = EM = MF = FN = NG = GO = OH = HP = PA$

From Figure 3.4，*ABCDEFGH* is a regular polygon

Therefore, $\angle ABC = \angle BCD$

$\angle IBJ = \angle ABC - \angle IBA - \angle JBC$

And $\angle JCK = \angle BCD - \angle JCB - \angle KCD$

$\therefore \angle IBJ = \angle JCK$

In $\triangle IBJ$ and $\triangle JCK$, $IB = JC$, $\angle IBJ = \angle JCK$, $BJ = CK$

Therefore, $\triangle IBJ \equiv \triangle JCK$（SAS）

Similarly, $\triangle IBJ \equiv \triangle JCK \equiv \triangle KDL \equiv \triangle LEM \equiv \triangle MFN \equiv \triangle NGO \equiv \triangle OHP \equiv \triangle PAI$

$\therefore IJ = JK = KL = LM = MN = NO = OP = PI$

$\therefore$ Octagon *IJKLMNOP* is a regular octagon.

In order to find the relationship between *RQ* and $\angle AIB$ & $\angle AQB$, we look at $\triangle AQI$ and $\triangle BQI$.

From these two triangles, we see that $AQ = BQ$, *IQ* is the common length and $IA = IB$

Hence $\triangle AQI \equiv \triangle BQI$（SSS）

$\therefore \angle AIQ = \angle BIQ$, $\angle AQI = \angle BQI$

Therefore we found the relationship that *IQ* is the bisector of both $\angle AIB$ and $\angle AQB$

As both $\triangle ABQ$ and $\triangle ABI$ *are* isosceles triangles, *IQ* is perpendicular to *AB.*

Hence $\angle ARQ = 90°$which proved that $\triangle ARQ$ is a right-angle triangle.

In order to find the relationship of *SQ* and *IP,* we refer to Figure 3.3. As we proven previously that octagon *IJKLMNOP* is a regular octagon, $PQ = IQ$.

Therefore $\triangle PQI$ is an isosceles triangle.

In this isosceles triangle, *AQ* bisectors $\angle PQI$

Hence, *QS* is perpendicular to *PI.*

Therefore $\triangle ISQ$ is a right-angle triangle.

In previous working, we found that both $\triangle ARQ$ and $\triangle IQS$ are right-angle triangles.

Here, we want to find the relationship between these two triangles.

In $\triangle AQR$ and $\triangle IQS$, $\angle AQR$ is the common angle and $\angle ARQ = \angle ISQ = 90°$

Therefore, $\triangle AQR$ is similar as $\triangle IQS$

Therefore $\frac{SI}{AR} = \frac{IQ}{AQ}$, and $\frac{SI}{AR}$ is actually the length ratio of constructed polygon and the original one.

In order to find $\frac{SI}{AR}$, we need to find the length of *IQ* and *AQ*.

Let length of IR be *x* units, and the number of sides of the polygon be *n.*

As $IQ = RQ - IR$, the length of *RQ* is required.

$\sin \angle AIR = \frac{AR}{IA}$

$AR = \sin 60° \bullet 2x$

Therefore， $AR = \sqrt{3}x$ **(3.6)**

$\angle AQB = \dfrac{360°}{n}$

$\therefore \angle AQR = \dfrac{1}{2} \angle AQB = \dfrac{180°}{n}$ **(3.7)**

In order to find the value of $\dfrac{IQ}{AQ}$, we need the length of *AQ*

$\sin \angle AQR = \dfrac{AR}{AQ}$

Substitute equations **3.1** and **3.3** into this equation.

$$AQ = \dfrac{\sqrt{3}x}{\sin \dfrac{180°}{n}} \quad \textbf{(3.8)}$$

We need to find the length of IQ. As *IQ* = *RQ* + *IR* and we let *IR* be *x,* the length of *RQ* is required.

$\because \tan \angle AQR = \dfrac{AR}{RQ}$

Substitute equations **3.1** and **3.2** into this equation.

$$RQ = \dfrac{\sqrt{3}x}{\tan \dfrac{180°}{n}} \quad \textbf{(3.9)}$$

In order to find the value of $\dfrac{IQ}{AQ}$, and find the length ratio between the constructed polygon and the original polygon. We need the length of *IQ* as we have obtained the length of *AQ* (refer to equation **3.3**)

In Figure 3.3, *QI* = *QR* — *RI*

By substituting equation **3.4** into *QI* = *QR* — *RI*.

$$QI = x - \frac{\sqrt{3}x}{\tan \frac{180°}{n}} \qquad \textbf{(3.10)}$$

As we got the value of *IQ* and *AQ* (refer to equations **3.8** and **3.10**), we can hence find the value of $\frac{SI}{AR}$.

Therefore, $\frac{SI}{AR} = \frac{IQ}{AQ}$

$$= \frac{\frac{\sqrt{3}x}{\tan \frac{180°}{n}} - x}{\frac{\sqrt{3}x}{\sin \frac{180°}{n}}}$$

$$= -\frac{\sqrt{3}}{3}\sin\frac{180°}{n} + \cos\frac{180°}{n}$$

Therefore, length ratio at sides of the constructed inner polygon to the original regular polygon

$= -\frac{\sqrt{3}}{3}\sin\frac{180°}{n} + \cos\frac{180°}{n}$.

3.5 Area ratio of the difference in area of two constructed regular polygon to the original regular polygon

Let the area of the original regular polygon be A units².

As we have found the length ratios at sides of the constructed outward and inward polygons and the original polygon, which are $\frac{\sqrt{3}}{3}\sin\frac{180°}{n}+\cos\frac{180°}{n}$ and $-\frac{\sqrt{3}}{3}\sin\frac{180°}{n}+\cos\frac{180°}{n}$ .The squares of these two ratios are the area ratio of the constructed outward and inward polygons respectively.

Therefore, area of constructed outer regular polygon= $(\frac{\sqrt{3}}{3}\sin\frac{180°}{n}+\cos\frac{180°}{n})^2 A$ units².

And the area of constructed inner regular polygon= $(-\frac{\sqrt{3}}{3}\sin\frac{180°}{n}+\cos\frac{180°}{n})^2 A$ units².

The difference in the area of these two polygons

$$= (\frac{\sqrt{3}}{3}\sin\frac{180°}{n}+\cos\frac{180°}{n})^2 A - (-\frac{\sqrt{3}}{3}\sin\frac{180°}{n}+\cos\frac{180°}{n})^2 A$$

$$= \left(\frac{4\sqrt{3}}{3}\sin\frac{180°}{n}\cos\frac{180°}{n}\right)A$$

$$= \left(\frac{2\sqrt{3}}{3}\sin\frac{360°}{n}\right)A \quad \text{units}^2$$

Therefore, area ratio of the difference area on two constructed regular polygon to the original regular polygon= $\left(\frac{2\sqrt{3}}{3}\sin\frac{360°}{n}\right)A$ units².

Chapter 4: Analysis of results

4.1 Analysis of results

We have constructed equilateral triangles outwardly and inwardly on n-sided polygons. Now, we have found out the relationship between length of the original polygon and constructed polygons (inward or outward), the relationship between the difference in area of two constructed polygons and the original polygon.

The length ratio at sides of the constructed outer polygon to the original regular polygon is

$$\frac{\sqrt{3}}{3}\sin\frac{180°}{n}+\cos\frac{180°}{n}.$$

The length ratio at sides of the constructed inner polygon to the original regular polygon is

$$-\frac{\sqrt{3}}{3}\sin\frac{180°}{n}+\cos\frac{180°}{n}.$$

In addition, the area ratio of the difference area on two constructed regular polygon to the original regular polygon is $\left(\frac{2\sqrt{3}}{3}\sin\frac{360°}{n}\right)A$ units².

4.2 Recommendation

This method we have come up with only takes the 2D polygons into consideration. We can explore this formula into 3D figures in the future.

Chapter 5: Conclusion

We studied Napoleon's Theorem which states that if equilateral triangles are constructed on the sides of any given triangle, either all outward, or all inward, the centers of those equilateral triangles themselves form an equilateral triangle. The triangle thus formed is called the Napoleon triangle (inner and outer). In addition, the difference in area of these two triangles equals the area of the original triangle. However we found that this theorem only applies on triangles. We were curious about whether Napoleon's Theorem works on n-sided figures or not.

In the project which we carried out last year, we constructed equilateral triangles outwardly on the sides of regular polygons (which we called the "Original Polygon"), and connected the centers of the equilateral triangles to form another regular polygon (which we called the "Constructed Polygon"). A formula connecting the lengths of the sides of the Original Polygon and the sides of Constructed Polygon was found. Another formula was found when we changed the equilateral triangles to regular polygons which had the same number of sides as the Original Polygon. However, due to the limitation of our knowledge and constrain of time, we were unable to prove these formulae vigorously and explore Napoleon's Theorem any further.

Hence, this year we started constructing equilateral triangles on the sides of a n-sided polygon (either inwardly or outwardly).

After the mathematics proves, we have found out that the length ratio at sides of the constructed outer polygon to the original regular polygon = $-\frac{\sqrt{3}}{3}\sin\frac{180°}{n}+\cos\frac{180°}{n}$.

Also the area ratio of the difference area on two constructed regular polygon to the original regular polygon = $\left(\frac{2\sqrt{3}}{3}\sin\frac{360°}{n}\right)A$ units^2.

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