The Extension of De Gua’s Theorem

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2021

Abstract

Out of the many theorems related to geometry, this report will be focusing on De Gua's Theorem. De Gua's Theorem is a three-dimensional analog of Pythagorean theorem. This theorem states that if a tetrahedron has a right-angle corner, then the square of the area of the face opposite the right-angle corner is the sum of the squares of the areas of the other three faces. However, this theorem can only apply to tetrahedrons with a right-angle corner, which means tetrahedrons with three right-angled triangles.

Our main objective is to generate a formula extended from De Gua's Theorem to be applied to tetrahedrons with less right-angled triangles. We have also successfully further extended the application of the formula to be used on *n*-sided-polygon-based pyramids.

Acknowledgement

We would like to thank our Mathematics teacher, Ms Kok Lai Fong, for guiding us through the whole journey of exploring and extending De Gua's Theorem. She kept our passion burning and encouraged us to go on when we hit a dead end. She had also spent countless times with us, listening to our presentation and giving us constructive feedbacks and advices. We are also very grateful to be given the chance to participate in this meaningful competition. This project had exposed us to Mathematics out of the classroom, and we had learnt a lot along the journey of discovering the beauty of math. We appreciate this opportunity and hope for similar competitions in the future.

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Chapter 1: Introduction

Among the various fields of Mathematics studies, we decided to do some research on geometry, specifically on 3D geometry, and we found De Gua's Theorem. De Gua's Theorem states that that if a tetrahedron has a right-angle corner, then the square of the area of the face opposite the right-angle corner is the sum of the squares of the areas of the other three faces. After researching more on this theorem, we found that De Gua's Theorem is defined to be applied on only tetrahedrons with a right-angle corner. Hence, we came out with an idea to extend De Gua's Theorem so that it can be applied to all types of tetrahedron. After that, we move on to pyramids with n-sided-polygon bases. For easier reference, in this report, we define the polygon which we are interested to find its area square as the base of the tetrahedrons or pyramids.

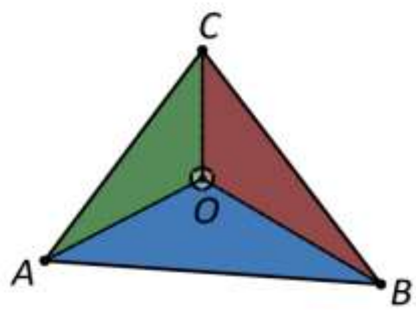
Our methodology uses Heron's Formula and other formulas to calculate the area of triangles (such as $\frac{1}{2}bh$ and $\frac{1}{2}ab\sin c$) to derive the extension of De Gua's Theorem for tetrahedrons. Then, by cutting n-sided-polygon-based pyramids into a few tetrahedrons, we can apply our own formula to these pyramids.

In this report, we focus mainly on the derivation of the extension of De Gua's Theorem and how we apply it on n-sided-polygon-based pyramids.

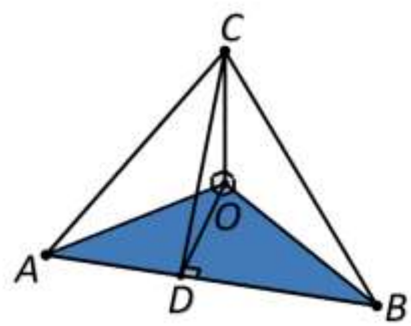
Chapter 2: Literature Review

A. De Gua's Theorem

We did some research and understood De Gua's Theorem. De Gua's Theorem states that that if a tetrahedron has a right-angle corner, then the square of the area of the face opposite the right-angle corner is the sum of the squares of the areas of the other three faces. To proof this theorem, given a tetrahedron *OABC* while *OC*, *OB* and *OA* are all at right angles to each other.



Put a point *D* on line *AB*, such that *OD* is perpendicular to *AB*.



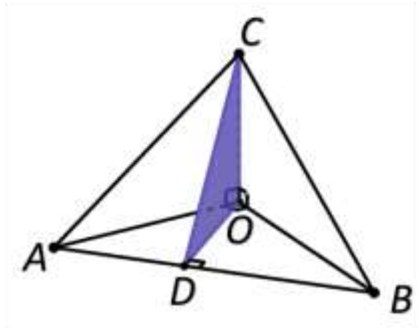
Area of triangle AOB = $\frac{1}{2} \times OA \times OB = \frac{1}{2} \times AB \times OD$

Squaring both sides,

$$\left(\frac{1}{2} \times OA \times OB\right)^2 = \left(\frac{1}{2} \times AB \times OD\right)^2$$

$$\frac{1}{4} \times OA^2 \times OB^2 = \frac{1}{4} \times AB^2 \times OD^2 \text{------------} (1)$$

Next, considering triangle *COD*,



By Pythagoras Theorem,

$$OD^2 = CD^2 - OC^2 \text{ ------------ } (2)$$

Substituting (2) into (1).

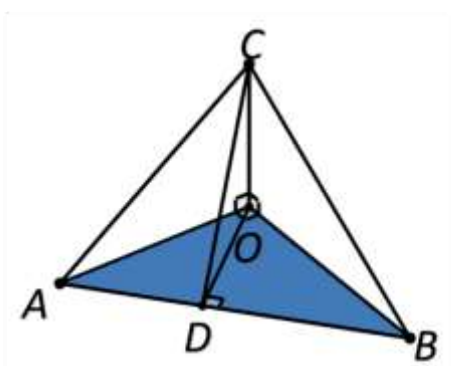
$$\frac{1}{4} \times OA^2 \times OB^2 = \frac{1}{4} \times AB^2 \times OD^2 \text{ ------------ } (1)$$

$$\frac{1}{4} \times OA^2 \times OB^2 = \frac{1}{4} \times AB^2 \times (CD^2 - OC^2)$$

$$\frac{1}{4} \times OA^2 \times OB^2 = \left(\frac{1}{4} \times AB^2 \times CD^2\right) - \left(\frac{1}{4} \times AB^2 \times OC^2\right)$$

$$\frac{1}{4} \times OA^2 \times OB^2 + \left(\frac{1}{4} \times AB^2 \times OC^2\right) = \left(\frac{1}{4} \times AB^2 \times CD^2\right) \text{ ------------- } (3)$$

Now considering triangle *AOB*,



By Pythagoras Theorem,

$$AB^2 = OA^2 + OB^2 \text{ -------------- } (4)$$

Substitute (4) into (3).

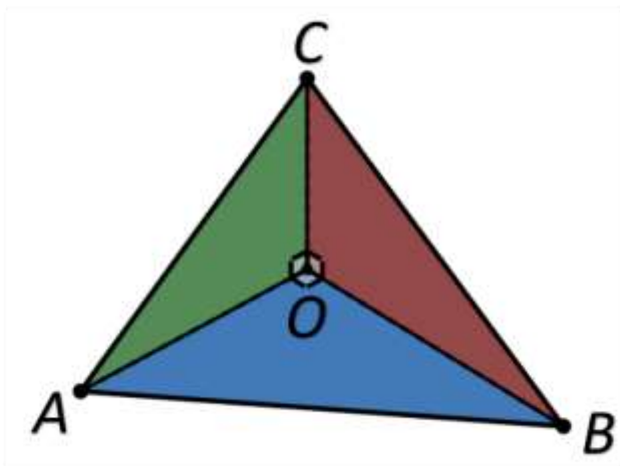
$$\frac{1}{4} \times OA^2 \times OB^2 + \left(\frac{1}{4} \times AB^2 \times OC^2\right) = \left(\frac{1}{4} \times AB^2 \times CD^2\right) \text{------------} (3)$$

$$AB^2 = OA^2 + OB^2 \text{--------------} (4)$$

$$\left(\frac{1}{4} \times OA^2 \times OB^2\right) + \left[\frac{1}{4} \times (OA^2 + OB^2) \times OC^2\right] = \left(\frac{1}{4} \times AB^2 \times CD^2\right)$$

$$\left(\frac{1}{4} \times OA^2 \times OB^2\right) + \left(\frac{1}{4} \times OA^2 \times OC^2\right) + \left(\frac{1}{4} \times OB^2 \times OC^2\right) = \left(\frac{1}{4} \times AB^2 \times CD^2\right)$$

$$\left(\frac{1}{2} \times OA \times OB\right)^2 + \left(\frac{1}{2} \times OA \times OC\right)^2 + \left(\frac{1}{2} \times OB \times OC\right)^2 = \left(\frac{1}{2} \times AB \times CD\right)^2$$



Looking back at tetrahedron *OABC*, we can rewrite this equation into

$$\left(A_{AOB}\right)^2 + (A_{AOC})^2 + \left(A_{BOC}\right)^2 = A_{ABC}^2$$

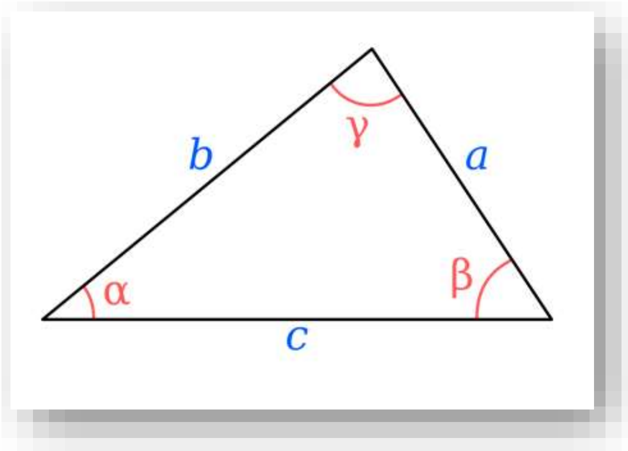
Hence, we can say that when a tetrahedron has a right-angle corner, then the square of the area of the face opposite the right-angle corner (triangle *ABC*) is the sum of the squares of the areas of the other three faces, which is stated by De Gua's Theorem.

B. <u>Heron's Formula</u>

We also did some research on Heron's Formula, another important formula that we apply to extend De Gua's Theorem. Heron's Formula gives the area of a triangle when the lengths of all three sides are known. Given a triangle with sides a, b and c, Heron's Formula states that

$$\text{Area of triangle} = \sqrt{s(s-a)(s-b)(s-c)}, \text{ where } s = \frac{a+b+c}{2}$$

To proof this theorem, given a triangle with sides a, b, c and angles α, β, γ.



By Cosine Rule,

$$\cos\gamma = \frac{a^2+b^2-c^2}{2ab} \text{ --------------------- } (1)$$

Using Trigo Identities:

$$\sin^2\gamma + \cos^2\gamma = 1$$

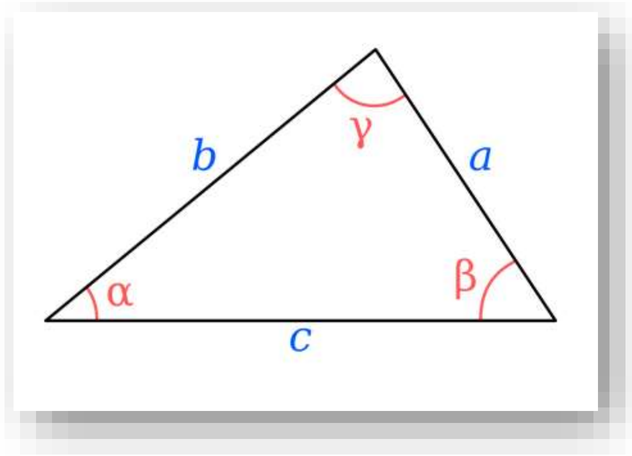
$$\sin\gamma = \sqrt{1-\cos^2\gamma} \text{ --------------------- } (2)$$

Substituting (1) into (2),

$$\sin\gamma = \sqrt{1-\left(\frac{a^2+b^2-c^2}{2ab}\right)^2}$$

$$\sin\gamma = \sqrt{\frac{4a^2b^2}{4a^2b^2} - \frac{(a^2+b^2-c^2)^2}{4a^2b^2}}$$

$$\sin\gamma = \frac{\sqrt{4a^2b^2-(a^2+b^2-c^2)^2}}{2ab} \text{ ------------------- } (3)$$



$Area\ of\ triangle = \frac{1}{2}ab\sin\gamma$ ---------------- (4)

$\sin\gamma = \frac{\sqrt{4a^2b^2-(a^2+b^2-c^2)^2}}{2ab}$ --------------- (3)

Substituting (3) into (4),

$Area\ of\ triangle$

$$= \frac{1}{2}ab\left[\frac{\sqrt{4a^2b^2-(a^2+b^2-c^2)^2}}{2ab}\right]$$

$$= \frac{1}{4}\sqrt{4a^2b^2-(a^2+b^2-c^2)^2}$$

$$= \frac{1}{4}\sqrt{[2ab-(a^2+b^2-c^2)][2ab+(a^2+b^2-c^2)]}$$

$$= \frac{1}{4}\sqrt{[c^2-(a-b)^2][(a+b)^2-c^2]}$$

$$= \sqrt{\frac{[c-(a-b)][c+(a-b)][(a+b)-c][(a+b)+c]}{16}}$$

$$= \sqrt{\frac{(b+c-a)}{2}\frac{(a+c-b)}{2}\frac{(a+b-c)}{2}\frac{(a+b+c)}{2}}$$

$$Area\ of\ triangle = \sqrt{\frac{(a+b+c)}{2}\frac{(b+c-a)}{2}\frac{(a+c-b)}{2}\frac{(a+b-c)}{2}} \quad ----------------(5)$$

Given $s = \frac{a+b+c}{2}$ ------------(6)

Rewriting,

$$(s-a) = \frac{a+b+c}{2} - a = \frac{b+c-a}{2} \quad -------------(7)$$

$$(s-b) = \frac{a+b+c}{2} - b = \frac{a+c-b}{2} \quad -------------(8)$$

$$(s-c) = \frac{a+b+c}{2} - c = \frac{a+b-c}{2} \quad -------------(9)$$

Substituting (6) (7) (8) (9) into (5).

Area of triangle

$$= \sqrt{\frac{(a+b+c)}{2}\frac{(b+c-a)}{2}\frac{(a+c-b)}{2}\frac{(a+b-c)}{2}} \quad -------------(5)$$

$= \sqrt{s(s-a)(s-b)(s-c)}$ (Q.E.D)

Chapter 3: Methodology

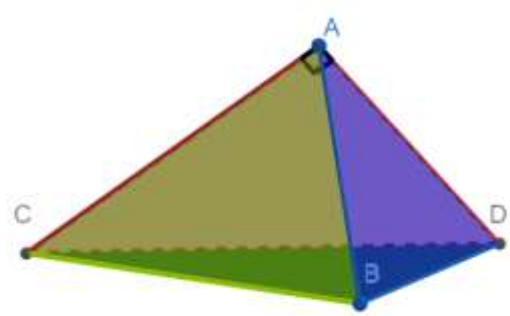
Our project on extending De Gua's Theorem is to generalize a formula to calculate the base area for pyramids with non-right-angled triangles. In order to accomplish our goal, we began with tetrahedrons, which is the 3D-model being used in De Gua's Theorem. This is due to the fact that we wanted to start with 3D-model that is applicable to original De Gua's theorem so that we can have a deeper insight on original De Gua's Theorem and to garner more inspirations along the way.

Considering original De Gua's Theorem**, this theorem is to find the square of area of triangle that is opposite the right-angled corner**. Nevertheless, keeping in mind our aim is to find square of area of triangle on tetrahedron with reduced number of right-angled triangles of tetrahedron, we reduced it to only one right-angled triangle needed. This is crucial as it serves as the reference point for us to determine which is the triangle face that we will have to find its area square (as that triangle which we will have to find its square area is opposite of the right corner), if not, we cannot determine which face of triangle to be our subject. It also helps to curb the floating variables and proved itself useful in our attempt to derive the extension of De Gua's Theorem.

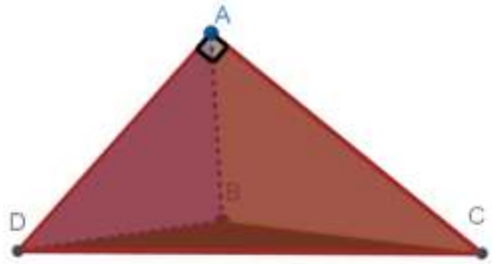
Study on tetrahedron with at least one right-angled triangle face

1. Construct a tetrahedron with one right-angled triangle face

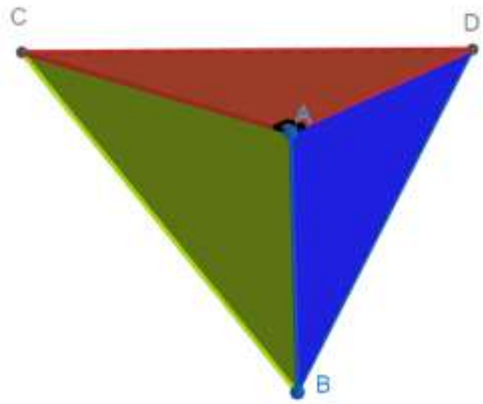
i. Front view of the tetrahedron:



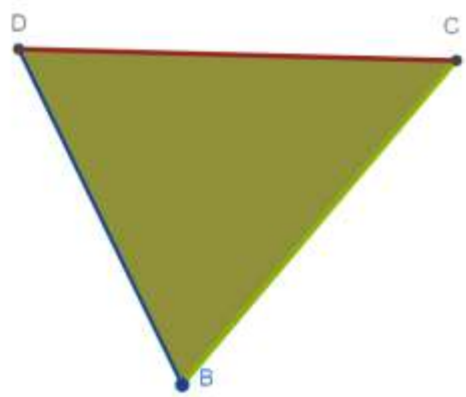
ii. Back view of the tetrahedron (Right triangle is represented by red triangle):



iii. Top view of the tetrahedron:

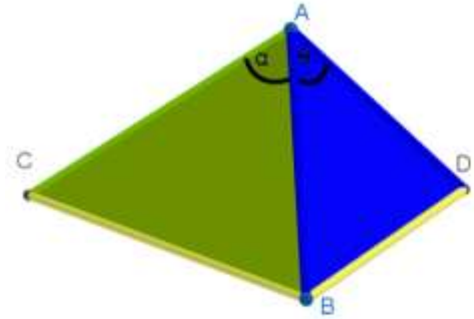


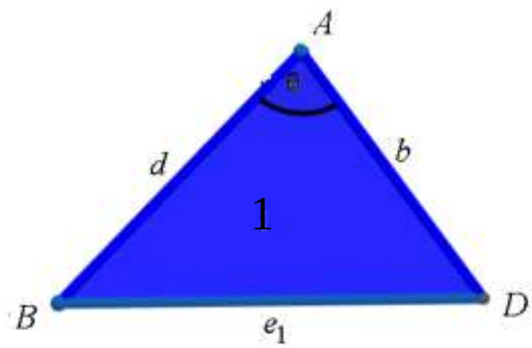
iv. Bottom view of the tetrahedron:



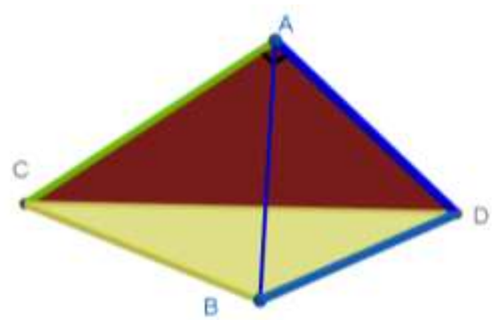
2. Label the different faces and side of tetrahedron

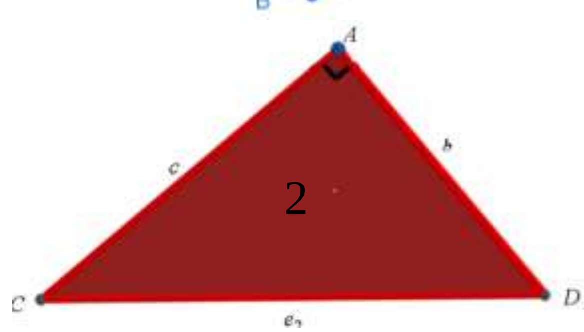
i. Label this blue triangle as triangle 1, and hence its area is $\boldsymbol{A_1}$



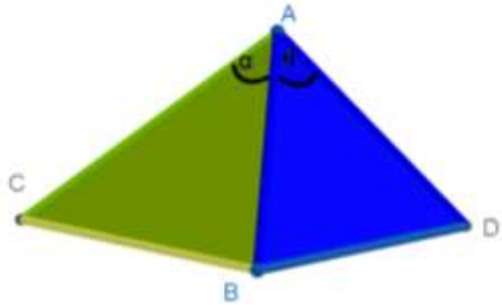


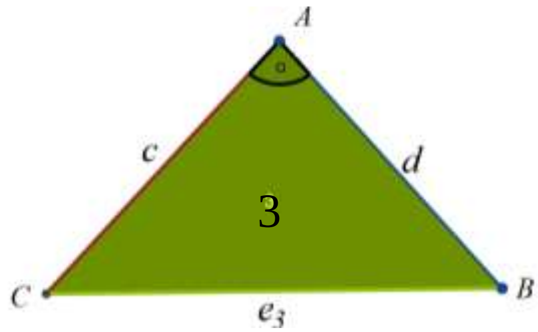
ii. Label this red triangle (right-angled triangle) as triangle 2, and hence its area is $\boldsymbol{A_2}$; triangles at the front are made transparent.



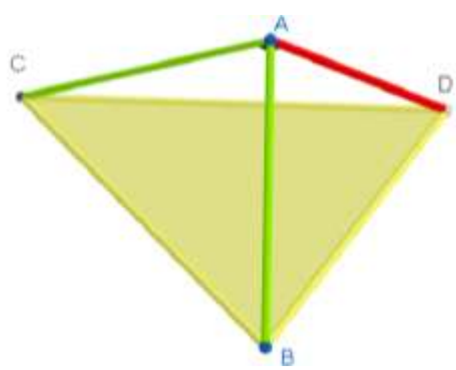


iii. Label this green triangle as triangle 3, and hence its area is $\boldsymbol{A_3}$

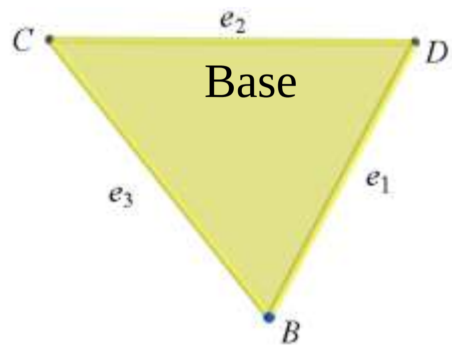




iv. Label this yellow triangle as base triangle, which is the triangle we are interested in finding the square of its area. This is because it is opposite of the right angle, and hence its area is $\boldsymbol{A_{base}}$. The triangular faces blocking the base triangle is being made transparent to have a better view of base triangle.



Derivation of the extension of De Gua's Theorem



Since we are interested in base triangle, the square of its area is the subject of the equation.

Using Heron's Formula,

$$A_{Base} = \sqrt{S(S-e_1)(S-e_2)(S-e_3)}, where\ \mathrm{S} = \frac{e_1 + e_2 + e_3}{2}$$

Square both sides,

$$(A_{Base})^2 = \left(\sqrt{S(S-e_1)(S-e_2)(S-e_3)}\right)^2$$

$$(A_{Base})^2 = S(S-e_1)(S-e_2)(S-e_3)\text{---------------------------------}(1)$$

Writing e_1, $e_2$and e_3 in terms of length of side of triangle 1, 2 and 3 and angles,

Using Cosine Rule,

$$(e_1)^2 = b^2 + d^2 - 2bd\cos\theta$$

$$e_1 = \sqrt{b^2 + d^2 - 2bd\cos\theta}$$

$$(e_3)^2 = c^2 + d^2 - 2cd\cos\alpha$$

$$e_3 = \sqrt{c^2 + d^2 - 2cd\ cos\alpha}$$

Using Pythagoras Theorem,

$$(e_2)^2 = b^2 + c^2$$

$$e_2 = \sqrt{b^2 + c^2}$$

Given S $= \frac{e_1 + e_2 + e_3}{2}$,

Substitute S and e_1, e_2, e_3 into (1),

$(A_{Base})^2 = S(S - e_1)(S - e_2)(S - e_3)$--------------------(1)

$$(A_{Base})^2 = \frac{e_1 + e_2 + e_3}{2}\left(\frac{e_1 + e_2 + e_3}{2} - \sqrt{b^2 + d^2 - 2bd\,cos\,\theta}\right) \left(\frac{e_1 + e_2 + e_3}{2} - \sqrt{b^2 + c^2}\right)\left(\frac{e_1 + e_2 + e_3}{2} - \sqrt{c^2 + d^2 - 2cd\,cos\alpha}\right)$$

Continue by substituting e_1, e_2, e_3 into S,

$$(A_{Base})^2 = \frac{\sqrt{b^2 + d^2 - 2bd\cos\theta} + \sqrt{b^2 + c^2} + \sqrt{c^2 + d^2 - 2cd\cos\alpha}}{2} \left(\frac{\sqrt{b^2 + d^2 - 2bd\cos\theta} + \sqrt{b^2 + c^2} + \sqrt{c^2 + d^2 - 2cd\cos\alpha}}{2} - \sqrt{b^2 + d^2 - 2bd\cos\theta}\right) \left(\frac{\sqrt{b^2 + d^2 - 2bd\cos\theta} + \sqrt{b^2 + c^2} + \sqrt{c^2 + d^2 - 2cd\cos\alpha}}{2} - \sqrt{b^2 + c^2}\right) \left(\frac{\sqrt{b^2 + d^2 - 2bd\cos\theta} + \sqrt{b^2 + c^2} + \sqrt{c^2 + d^2 - 2cd\cos\alpha}}{2} - \sqrt{c^2 + d^2 - 2cd\cos\alpha}\right)$$

...

$$(A_{Base})^2 = \frac{b^2c^2 + b^2d^2 + c^2d^2 - 2bdc^2\cos\theta - 2b^2cd\cos\alpha - c^2d^2\cos^2\alpha + 2bcd^2\cos\alpha\cos\theta - bd^2\cos^2\theta}{4}$$

$$= \frac{b^2c^2}{4} + \frac{b^2d^2}{4} + \frac{c^2d^2}{4} - \frac{2bdc^2\cos\theta}{4} - \frac{2b^2cd\cos\alpha}{4} - \frac{c^2d^2\cos^2\alpha}{4} + \frac{2bcd^2\cos\alpha\cos\theta}{4} - \frac{bd^2\cos^2\theta}{4}$$

$$= \left(\frac{bc}{2}\right)^2 + \left(\frac{b^2d^2}{4}\right) + \left(\frac{c^2d^2}{4}\right) - c^2\left(\frac{bd\cos\theta}{2}\right) - b^2\left(\frac{cd\cos\alpha}{2}\right) - \left(\frac{cd\cos\alpha}{2}\right)^2 + 2\left[\left(\frac{bd\cos\theta}{2}\right)\left(\frac{cd\cos\alpha}{2}\right)\right] - \left(\frac{bd\cos\theta}{2}\right)^2$$

Using trigonometric identity,
$\cos^2\theta + \sin^2\theta = 1,$
$\therefore \cos\theta = \sqrt{1-\sin^2\theta}$ & $\cos\alpha = \sqrt{1-\sin^2\alpha}$

Substituting $\cos\theta = \sqrt{1-\sin^2\theta}$ & $\cos\alpha = \sqrt{1-\sin^2\alpha}$,

$$(A_{Base})^2$$

$$= \left(\frac{bc}{2}\right)^2 + \left(\frac{b^2d^2}{4}\right) + \left(\frac{c^2d^2}{4}\right) - c^2\left(\frac{bd\cos\theta}{2}\right) - b^2\left(\frac{cd\cos\alpha}{2}\right) - \left(\frac{cd\cos\alpha}{2}\right)^2$$

$$+2\left[\left(\frac{bd\cos\theta}{2}\right)\left(\frac{cd\cos\alpha}{2}\right)\right] - \left(\frac{bd\cos\theta}{2}\right)^2$$

$$= \left(\frac{bc}{2}\right)^2 + \left(\frac{b^2d^2}{4}\right) + \left(\frac{c^2d^2}{4}\right) - c^2\left(\frac{bd\sqrt{1-\sin^2\theta}}{2}\right) - b^2\left(\frac{cd\sqrt{1-\sin^2\alpha}}{2}\right) - \left(\frac{cd\sqrt{1-\sin^2\alpha}}{2}\right)^2 +$$

$$2\left[\left(\frac{bd\sqrt{1-\sin^2\theta}}{2}\right)\left(\frac{cd\sqrt{1-\sin^2\alpha}}{2}\right)\right] - \left(\frac{bd\sqrt{1-\sin^2\theta}}{2}\right)^2$$

$$= \left(\frac{bc}{2}\right)^2 + \left(\frac{b^2d^2}{4}\right) + \left(\frac{c^2d^2}{4}\right) - c^2\left(\frac{bd\sqrt{1-\sin^2\theta}}{2}\right) - b^2\left(\frac{cd\sqrt{1-\sin^2\alpha}}{2}\right) - \frac{c^2d^2}{4} + \frac{c^2d^2\sin^2\alpha}{4} +$$

$$2\left[\left(\frac{bd\sqrt{1-\sin^2\theta}}{2}\right)\left(\frac{cd\sqrt{1-\sin^2\alpha}}{2}\right)\right] - \frac{b^2d^2}{4} + \frac{b^2d^2\sin^2\theta}{4}$$

$$= \left(\frac{bc}{2}\right)^2 - c^2\left(\frac{bd\sqrt{1-\sin^2\theta}}{2}\right) - b^2\left(\frac{cd\sqrt{1-\sin^2\alpha}}{2}\right) + \frac{c^2d^2\sin^2\alpha}{4} + 2\left[\left(\frac{bd\sqrt{1-\sin^2\theta}}{2}\right)\left(\frac{cd\sqrt{1-\sin^2\alpha}}{2}\right)\right] +$$

$$\frac{b^2d^2\sin^2\theta}{4}$$

Rewriting $\left(\frac{\boldsymbol{bd\sqrt{1-sin^2\theta}}}{\mathbf{2}}\right)$,

$$\left(\frac{bd\sqrt{1-\sin^2\theta}}{2}\right)$$

$$= \sqrt{\left(\frac{bd\sqrt{1-\sin^2\theta}}{2}\right)^2}$$

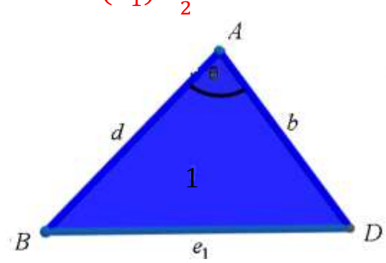
$$= \sqrt{\frac{b^2d^2(1-\sin^2\theta)}{4}}$$

$$= \sqrt{\frac{b^2d^2}{4} - \frac{b^2d^2\sin^2\theta}{4}}$$

$$= \sqrt{\frac{b^2d^2}{4} - \left(\frac{1}{2}bd\sin\theta\right)^2}$$

$$= \sqrt{\left(\frac{bd}{2}\right)^2 - (A_1)^2}$$

$(A_1) = \frac{1}{2}bd\sin\theta$



Rewriting $\left(\frac{cd\sqrt{1-\sin^2\alpha}}{2}\right)$,

$$\left(\frac{cd\sqrt{1-\sin^2\alpha}}{2}\right)$$

$$= \sqrt{\left(\frac{cd\sqrt{1-\sin^2\alpha}}{2}\right)^2}$$

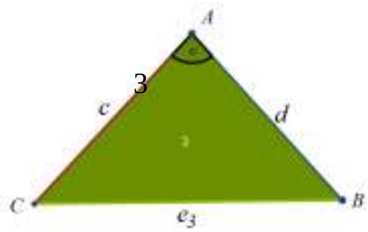
$$= \sqrt{\frac{c^2d^2(1-\sin^2\alpha)}{4}}$$

$$= \sqrt{\frac{c^2d^2}{4} - \frac{c^2d^2\sin^2\alpha}{4}}$$

$$= \sqrt{\frac{c^2d^2}{4} - \left(\frac{1}{2}cd\sin\alpha\right)^2}$$

$$= \sqrt{\left(\frac{cd}{2}\right)^2 - (A_3)^2}$$

$(A_3) = \frac{1}{2}cd\sin\alpha$



Hence,

$$(A_{Base})^2$$

$$=\left(\frac{bc}{2}\right)^2 - c^2\left(\frac{bd\sqrt{1-\sin^2\theta}}{2}\right) - b^2\left(\frac{cd\sqrt{1-\sin^2\alpha}}{2}\right) + \frac{c^2d^2\sin^2\alpha}{4} + 2\left[\left(\frac{bd\sqrt{1-\sin^2\theta}}{2}\right)\left(\frac{cd\sqrt{1-\sin^2\alpha}}{2}\right)\right] + \frac{b^2d^2\sin^2\theta}{4}$$

$$=\left(\frac{bc}{2}\right)^2 - c^2\left(\sqrt{(\frac{bd}{2})^2 - (A_1)^2}\right) - b^2\left(\sqrt{(\frac{cd}{2})^2 - (A_3)^2}\right) + \frac{c^2d^2\sin^2\alpha}{4} + 2\left[\left(\sqrt{(\frac{bd}{2})^2 - (A_1)^2}\right)\left(\sqrt{(\frac{cd}{2})^2 - (A_3)^2}\right)\right] + \frac{b^2d^2\sin^2\theta}{4}$$

$$= \frac{b^2d^2\sin^2\theta}{4} + \left(\frac{1}{2}bc\right)^2 + \left(\frac{c^2d^2\sin^2\alpha}{4}\right)^2 - c^2\left(\sqrt{\left(\frac{bd}{2}\right)^2 - (A_1)^2}\right) + 2\left[\left(\sqrt{\left(\frac{bd}{2}\right)^2 - (A_1)^2}\right)\left(\sqrt{\left(\frac{cd}{2}\right)^2 - (A_3)^2}\right)\right] - b^2\left(\sqrt{\left(\frac{cd}{2}\right)^2 - (A_3)^2}\right)$$

$$= \left(\frac{1}{2}bd\sin\theta\right)^2 + \left(\frac{1}{2}bc\right)^2 + \left(\frac{1}{2}cd\sin\alpha\right)^2 - c^2\left(\sqrt{\left(\frac{bd}{2}\right)^2 - (A_1)^2}\right) +$$

$$2\left[\left(\sqrt{\left(\frac{bd}{2}\right)^2 - (A_1)^2}\right)\left(\sqrt{\left(\frac{cd}{2}\right)^2 - (A_3)^2}\right)\right] - b^2\left(\sqrt{\left(\frac{cd}{2}\right)^2 - (A_3)^2}\right)$$

Given
$A_1 = \frac{1}{2}bd\sin\theta$
$A_2 = \frac{1}{2}bc$
$A_3 = \frac{1}{2}cd\sin\alpha$

Substitute A_1, A_2 , *and* A_3 ,

$$\therefore (A_{Base})^2 = (A_1)^2 + (A_2)^2 + (A_3)^2 - c^2\left(\sqrt{\left(\frac{bd}{2}\right)^2 - (A_1)^2}\right) +$$

$$2\left[\left(\sqrt{\left(\frac{bd}{2}\right)^2 - (A_1)^2}\right)\left(\sqrt{\left(\frac{cd}{2}\right)^2 - (A_3)^2}\right)\right] - b^2\left(\sqrt{\left(\frac{cd}{2}\right)^2 - (A_3)^2}\right)$$

Therefore, we have derived the extension of De Gua's Theorem with only area of triangles needed and side lengths of triangle. Our extension of De Gua's Theorem does not need any angle.

Chapter 4: Results & Analysis

A. Tetrahedrons

After deriving the extension of De Gua's Theorem, this is the generalized formula to calculate the area square of the base triangle of any types of tetrahedrons, be it tetrahedron with 1 ,2 or 3 right-angled triangles, as long as there is at least one-right-angled triangle for us to know the position of base triangle.

$$(A_{Base})^2 = (A_1)^2 + (A_2)^2 + (A_3)^2 - c^2\left(\sqrt{\left(\frac{bd}{2}\right)^2 - (A_1)^2}\right) + 2\left[\left(\sqrt{\left(\frac{bd}{2}\right)^2 - (A_1)^2}\right)\left(\sqrt{\left(\frac{cd}{2}\right)^2 - (A_3)^2}\right)\right] - b^2\left(\sqrt{\left(\frac{cd}{2}\right)^2 - (A_3)^2}\right)$$

Application of the extension of De Gua's Theorem

$$(A_{Base})^2 = (A_1)^2 + (A_2)^2 + (A_3)^2 - c^2\left(\sqrt{\left(\frac{bd}{2}\right)^2 - (A_1)^2}\right) + 2\left[\left(\sqrt{\left(\frac{bd}{2}\right)^2 - (A_1)^2}\right)\left(\sqrt{\left(\frac{cd}{2}\right)^2 - (A_3)^2}\right)\right] - b^2\left(\sqrt{\left(\frac{cd}{2}\right)^2 - (A_3)^2}\right)$$

1. c and d are the length of side of **any one right-angled triangle** in the tetrahedron, but not the hypotenuse.
2. b and d are the length of side of triangle 1, but not the length of side that shared with base triangle.
3. c and d are the length of side of triangle 3, but not the side that shared with base triangle.
4. The extension of De Gua's Theorem has original De Gua's Theorem $(A_1)^2+(A_2)^2+(A_3)^2$, followed by a string of extension.
5. c^2will be multiplied term $\left(\sqrt{\left(\frac{bd}{2}\right)^2 - (A_1)^2}\right)$,which is the term including length of side and its area of triangle face opposite side c .
6. Similarly, b^2 is multiplied by $\left(\sqrt{\left(\frac{cd}{2}\right)^2 - (A_3)^2}\right)$, which is the term including length of side and its area of triangle face opposite side b.

B. N-Sided-Polygon-Based Pyramids

After deriving the extension of De Gua's Theorem, we want to apply it onto n-sided-polygon-base pyramids. We have found this method of applying the extension of De Gua's Theorem onto an n-sided-polygon-base pyramid:

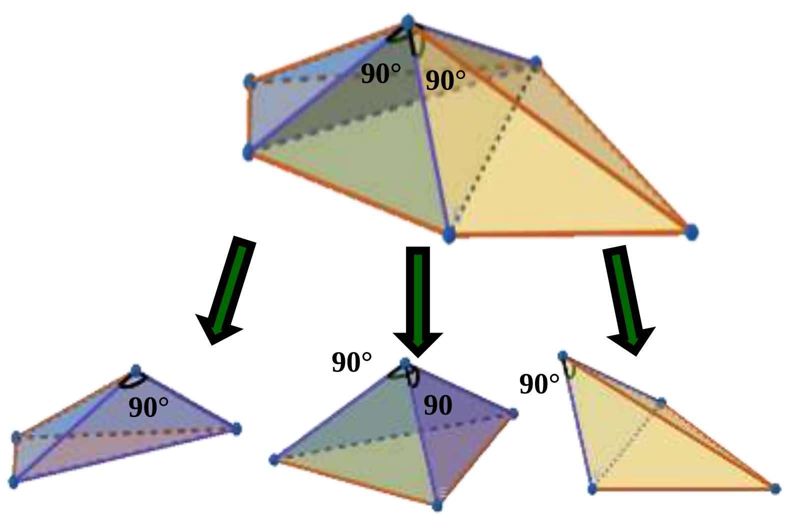
Step 1: Separate the pyramid into few tetrahedrons.

a) In order to separate the pyramid into few tetrahedrons, the first thing we have to do is to separate the base polygon into few triangles. The number of triangles separated follows the (n-2) sequence. For example, the number of triangles for pentagon base is 3 and for hexagon is 4. After that, the tetrahedrons will be separated following the triangles as base. Hence, the number of tetrahedrons also follows the (n-2) sequence.

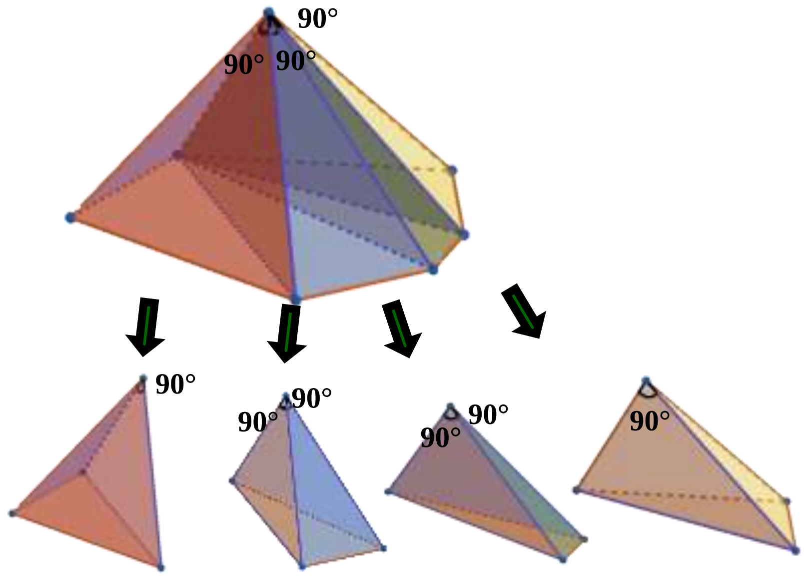
b) According to our extension of De Gua's Theorem, the base of the tetrahedron is always opposite the right angle, thus there must be at least 1 right-angled triangle in order to identify the base of the tetrahedron. Therefore, when separating the pyramid into few tetrahedrons, those tetrahedrons must have at least 1 right-angled triangle. Moreover, do also note that the tetrahedrons can share the same right-angled-triangular face.

c) These are the case examples of separating a pyramid into few tetrahedrons:

i. When the pyramid has pentagon base:

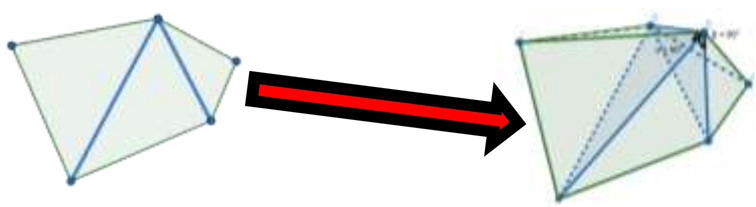


ii. When the pyramid has hexagon base:

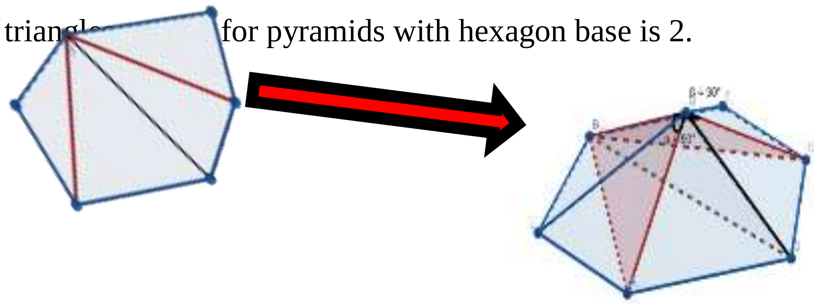


d) Additionally, since the tetrahedrons can share the same right-angled triangular face in some cases, we are curious to know the method to find the least number of right-angled triangles needed. Hence, we have observed that the number of right-angled triangles needed follows the following sequence:

i. For pyramids with odd number sided polygon base, least number of right-angled triangles needed follows $\left(\frac{n-1}{2}\right)$ sequence. For example, the least number of right-angled triangles needed for pyramids with pentagon base is 2.



ii. For pyramids with even number sided polygon base, least number of right-angled triangles needed follows $\left(\frac{n-2}{2}\right)$ sequence. For example, the least number of right-angled triangles needed for pyramids with hexagon base is 2.



Step 2: Apply the extension of De Gua's Theorem :

$$(A_{Base})^2$$

$$= (A_1)^2 + (A_2)^2 + (A_3)^2 - c^2\left(\sqrt{\left(\frac{bd}{2}\right)^2 - (A_1)^2}\right)$$

$$+ 2\left[\left(\sqrt{\left(\frac{bd}{2}\right)^2 - (A_1)^2}\right)\left(\sqrt{\left(\frac{cd}{2}\right)^2 - (A_3)^2}\right)\right] - b^2\left(\sqrt{\left(\frac{cd}{2}\right)^2 - (A_3)^2}\right)$$

to calculate the base area of the tetrahedrons separately, repeat this method (n-2) times as there are (*n*-2) tetrahedrons.

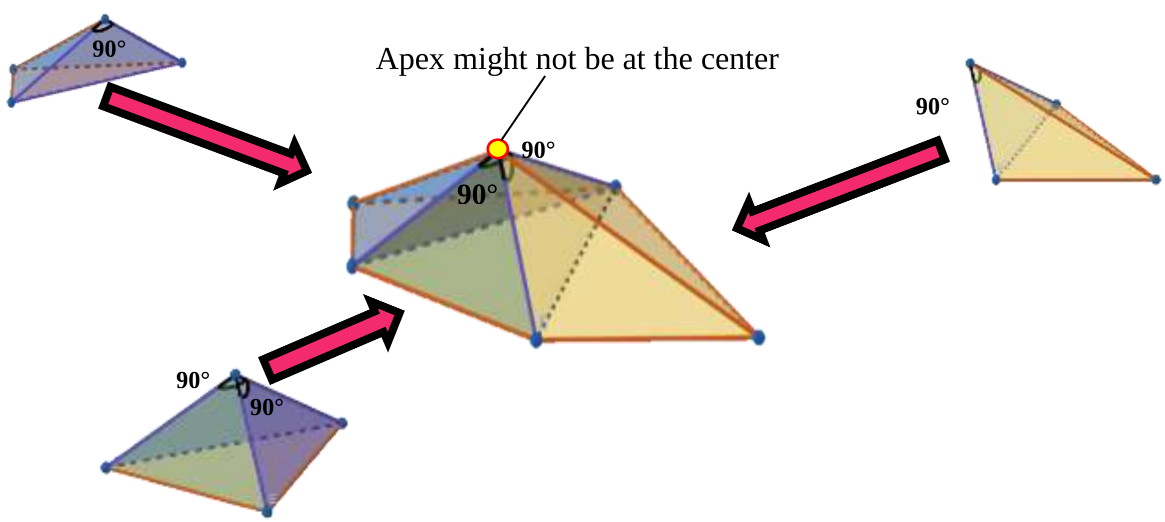
Step 3: Add up all the triangular base area of the (*n*-2) tetrahedrons.

After completing the three steps above, the total triangular base area of the (*n*-2) tetrahedrons will be the base area of the *n*-sided-polygon-based pyramids. Hence, this shows that the extension of De Gua's Theorem can be applied onto *n*-sided-polygon-base pyramids.

When applying the extension of De Gua's Theoren onto n-sided-polygon-base pyramids, there are some conditions that we have to take note:

1. According to extension of De Gua's Theorem, there must be at least 1 right-angled triangle in a tetrahedron in order to identify the base of the tetrahedron.
2. In the view of the fact that there must be be at least 1 right-angled triangle in a tetrahedron, we have to piece up tetrahedrons with 1 right-angled triangle to form a pyramid.
3. When we piece up the tetrahedrons into a pyramid, the apex of the pyramid might not be at the centre.

This is the example that depicts the aforementioned conditions when applying the extension of De Gua's Theoren onto *n*-sided-polygon-base pyramids:



Chapter 5: Conclusion

We have extended De Gua's Theorem and generalized a formula to calculate the base area for pyramids with non-right-angled triangles. After that, we compare De Gua's Theorem with our extension of De Gua's Theorem. The table below shows the comparison:

De Gua's Theorem	Our Extension of De Gua's Theorem
To use De Gua's Theorem, there must be 3 sides of right-angled triangle that form a right-angled corner.	To use our extension of De Gua's Theorem, we need only 1 right-angled triangle.
De Gua's Theorem is only applied to tetrahedrons with 1 right-angled corner.	Our extension of De Gua's Theorem can be applied to pyramids with n-sided polygon base.

To conclude, we have extended the De Gua's Theorem in which it can be more widely used and most importantly, we meet our objective in generating a formula extended from De Gua's Theorem to be applied to tetrahedrons with less right-angled triangles and extending the application of the formula so that the formula can be used on n-sided-polygon-based pyramids.

Reference

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