VIVIANI'S THEOREM AND RELATED PROBLEMS

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ABSTRACT

Out of the many theorems related to geometry, this report will be focusing on the Viviani's Theorem and its related problems. The theorem states that the sum of distances from any point inside an equilateral triangle is constant and is equal to the triangle's height. Any polygon that possesses constant sum of distances from an arbitrary interior point to the sides is said to have CVS *(constant V(x) sum)* property.

The two main objectives of the project are to approach Viviani's theorem and its extension using vectors and to establish a relationship linking the results of the earlier approach with Carnot's theorem - another theorem dealing with polygons.

From the first discovery, it was found that in any polygons with CVS property, the sum of unit vectors that are perpendicular to the respective sides of the polygons is a zero vector. This key finding helped clarify the rationale behind the result yielded in some earlier studies and helped further investigate some special geometrical properties to understand why certain polygons have CVS property. In the second discovery, Carnot's theorem was further extended to create a clear connection between the two theorems, Carnot's and Viviani's, due to the significant similarity in the two models observed.

INTRODUCTION

In the field of geometry, triangles are intriguing shapes thanks to their being an important starting point to explore polygons. Among various studies and theorems related to triangles, Viviani's Theorem is a study we find potential as it features basic conditions, thus, there would be opportunities for us to further study and investigate.

Viviani's Theorem is established by Italian mathematician Vincenzo Viviani (1622 – 1703). It states: "*In an equilateral triangle, the sum of distance from an arbitrary point to the three sides is always constant, and equal to the height of the triangle*."

The theorem has been proven, explored and extended in several ways: proof of converse theorem, extension to general triangles, extension to equilateral polygons, extension to polyhedra, etc, via numerous methods: area of triangle, coordinate geometry, linear programming and analysis geometry. However, so far we have only encountered two studies using the approach of vectors, one being a proof by Hans Samelson (2003) that generalises the theorem to equiangular polygons, and the other being a proof for converse theorem by Zhibo Chen and Tian Liang Chen (2006).

Vectors are a powerful tool in Mathematics, especially in Geometry because it works with both two-dimensional and three-dimensional space and can be algebraized despite being a geometrical tool.

In this report, a distance-sum function $V(x)$ is defined as the sum of distance from interior arbitrary point P in the given polygon to sides of the polygon. A polygon is said to possess CVS (constant $V(x)$ sum) property if the polygon has constant sum of distance from an arbitrary interior point to its sides.

LITERATURE REVIEW

Viviani's theorem is an easily-understood theorem for equilateral triangles. With it having a simple condition and significant result, the theorem has gained much attention from mathematicians to further study and extend the original statement. To gain a thorough insight of what has been discovered and achieved in the light of Viviani's Theorem, previous work are studied and investigated in this section.

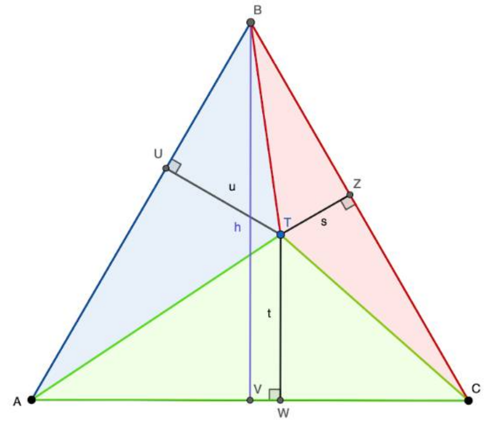
I. Original Viviani's Theorem for Equilateral Triangle:

Proposed by Vincenzo Viviani, a famous Italian Mathematician, Viviani's theorem stated:

> *Theorem 1. For any interior point P of an equilateral triangle ABC, the sum of the distances from P to the sides of $\triangle ABC$ is constant and equals to the height of $\triangle ABC$:* $\boldsymbol{s + u + t = h}$

Proof

The core idea of the proof is to find the relationship between the area and the height of the triangle. Given ABC is an equilateral triangle whose height is h and whose side is a. P is an arbitrary point in the given triangle. Let s, t and u be the distances of P to the sides of the triangle. Construct lines from P to A, B, and C to form triangles PCA, PBC and PAB whose areas are $\frac{ua}{2}, \frac{sa}{2}$ and $\frac{ta}{2}$ respectively.



Hence: $$\frac{ua}{2} + \frac{sa}{2} + \frac{ta}{2} = \frac{ha}{2}$$

Then, we can conclude that: $$u + s + t = h$$

II. Extension of Viviani's Theorem:

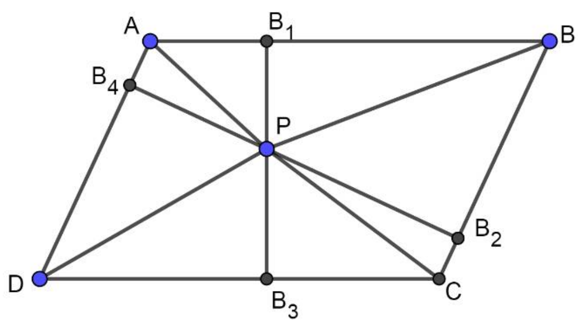
1. Parallelogram

The theorem was extended from the original Viviani's theorem for equilateral triangle to parallelogram (Zhibo Chen, Tian Liang 2006). The extension proposed:

Theorem 2. The sum of the distances from any point P inside a parallelogram is independent of the location of P.

Proof using area formula

Let S denote area. It holds:
$$S_{APB} = \frac{1}{2} PB_1 \times AB$$
$$S_{PBC} = \frac{1}{2} PB_2 \times BC$$
$$S_{APB} + S_{PBC} = \frac{1}{2} AB \times (PB_1 + PB_3)$$
$$PB_1 + PB_3 = \frac{S_{APB} + S_{PBC}}{\frac{1}{2}AB} = \frac{\frac{1}{2}\,area\ of\ the\ parallelogram}{\frac{1}{2}AB} = \frac{area\ of\ the\ parallelogram}{AB}$$



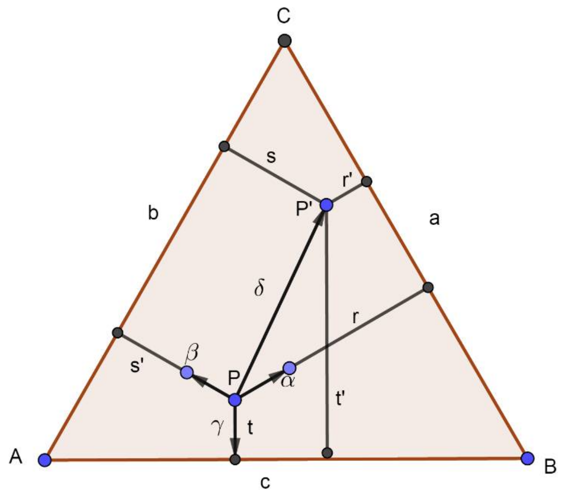
Similarly:
$$S_{PCD} = \frac{1}{2} PB_3 \times CD$$
$$S_{APD} = \frac{1}{2} PB_4 \times AD$$
$$PB_1 + PB_3 = \frac{area\ of\ the\ parallelogram}{BC}$$

$$PB_1 + PB_2 + PB_3 + PB_4 = area\ of\ the\ paralleolgram \times (\frac{1}{BC} + \frac{1}{AB})$$

Since the left-hand-side expression is a constant, the sum of distance is aso a constant. Thus, it is independent of the location of point P.

Proof without words using vectors (Hans Samelson, 2003)

$$|\alpha| = |\beta| = |\gamma|$$
$$\alpha + \beta + \gamma = 0$$
$$\alpha.\delta + \beta.\delta + \gamma.\delta = 0$$
$$(r - r') + (s - s') + (t - t') = 0$$
$$r + s + t = r' + s' + t'$$



The converse also holds:

Theorem 2.2. *If the sum of the distances from a point in the interior of a quadrilateral to the sides is independent of the location of the point, then the quadrilateral is a parallelogram.*

2. 2k-sided polygons with parallel opposite sides

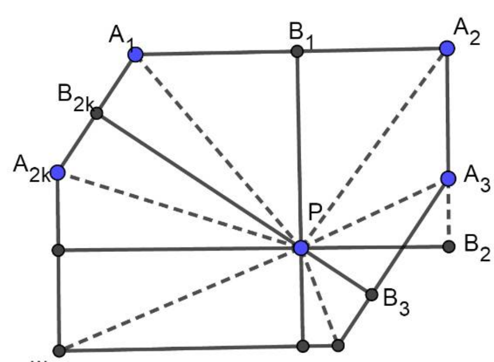
From the result above on the parallelogram, it is generalized to $2n$-gon with opposite sides parallel. In other words, it is stated:

Theorem 3. *In a 2k-sided polygon with opposite sides parallel, the sum of distances from an arbitrary point inside the polygon to its sides is constant and is independent of the point's position.*

Let P be the arbitrary interior point in the polygon. Let the opposite parallel side of A_iA_{i+1} be A_jA_{j+1}.

Since the polygon has *2k* sides, which is an even number and opposite sites are parallel, those pairs of opposite sides parallel would also have the same length.

Let the length of the first pair of opposite sides parallel be x_1.



$$S_{PA_iA_{i+1}} = \frac{1}{2}x_1(PB_i)$$

$$S_{PA_jA_{j+1}} = \frac{1}{2}x_1(PB_j)$$

$$S_{PA_iA_{i+1}} + S_{PA_jA_{j+1}} = \frac{1}{2}x_1(PB_j + PB_i)$$

Noted that the sum of all the triangles with the same vertex P equals to the area of the polygon and $PB_j + PB_i$ is indeed the distance between the pair of parallel sides. Since the sum of distances between any pair of opposite parallel sides is constant, it follows that the sum of all pairwise sums between the pairs of parallel sides, is also constant. However, the converse of this generalization is not true. A counterexample that can be easily found is an *equilateral hexagon*, which does not necessarily have opposite sides parallel but still have constant sum of distances from an arbitrary point.

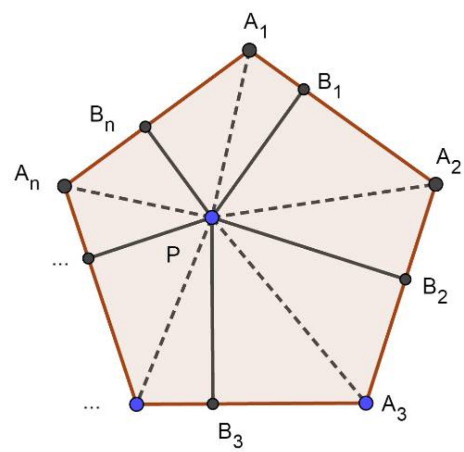
3. Regular polygons

The extension of Viviani's theorem in regular polygons states that:

Theorem 4. In every regular polygon, the sum of distances from an arbitrary point inside the polygon to its sides is constant and is independent of the point's position.

The most common method to prove the extensions related to Viviani's theorem is to use the method of area sum.

Let S denote area. It holds:



$$S_{PA_1A_2} = \frac{1}{2}PB_1(A_1A_2)$$

$$S_{PA_2A_3} = \frac{1}{2}PB_2(A_2A_3)$$

$$\dots$$

$$S_{PA_iA_{i+1}} = \frac{1}{2}PB_i(A_iA_{i+1})$$

$$\sum_{i=1}^{n} S_{PA_iA_{i+1}} = \frac{1}{2} \times \alpha \times \left(\sum_{i=1}^{n} PB_i\right)$$

Noted that:

$\sum_{i=1}^{n} S_{PA_iA_{i+1}} =$ *area of the polygon*

And α is the length of sides of the polygon

Thus: $\sum_{i=1}^{n} PB_i = \dfrac{S_{polygon}}{\frac{1}{2}\alpha}$

The converse of Viviani's extended theorem for regular polygon, however, is not true. The converse of this Viviani's theorem is stated as: *"If the sum of distance of an arbitrary interior point P of a given polygon is a constant and independent of the location of that point, the given polygon is regular."* The statement can be proved to be invalid by the counterexample of CVS property of parallelogram, which is not a regular polygon, in *Lit.rev 2.1.*

4. Equiangular polygons

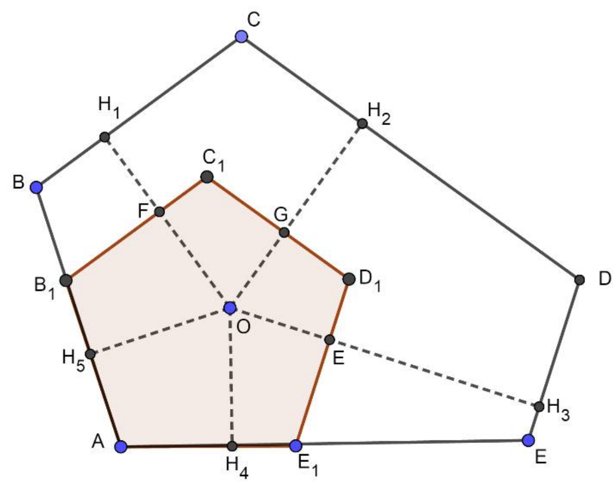
The Viviani's theorem holds not only for regular polygons but also for equiangular polygons. The extension, when reducing the condition of the polygon from "regular" to "equiangular" becomes more relaxed compared to the previous results. It states:

Theorem 5. The sum of distances from a point to the side lines of an equiangular polygon does not depend on the point and is that polygon's invariant.

In "On Viviani's Theorem and its Extensions" (p11, Example 4.6), Elias Abboud describes a proof for the extension above by locating inside the equiangular polygon a regular *n*-gon using parallel lines constructed. Here, the proof is represented.

Proof 1 (Elias Abboud. 2009)

Let equiangular *n*-gon *ABCDE* be $\mathcal{H}$.



Locate inside $\mathcal{H}$ a regular *n*-gon, $\mathcal{F}$. Rotate $\mathcal{F}$ around its centroid until one of its sides is parallel to one side of $\mathcal{H}$. As both polygons have the same number of sides, the included angle in-between any two adjacent sides in each polygon is fixed. Hence, when one side of $\mathcal{F}$ is parallel to one side of $\mathcal{H}$, all corresponding sides of both polygons are parallel.

Let $V_{\mathcal{H}}$ be the function of the sum of distance from point O to sides of polygon $\mathcal{H}$ and $V_{\mathcal{F}}$ be the function of the sum of distance from point $\mathcal{F}$ to sides of polygon $\mathcal{F}$:

$$V_{\mathcal{H}} = V_{\mathcal{F}} + FH_1 + GH_2 + IH_3$$

Because the area and lengths of the two polygons are fixed by condition, distances between its corresponding sides FH_1, GH_2, IH_3 are constant. Thus, for any point O inside $\mathcal{F}$ we have,

$$V_{\mathcal{H}} = V_{\mathcal{F}} + c,$$

where c represents the sum of distances between the parallel sides of $\mathcal{H}$ and $\mathcal{F}$.

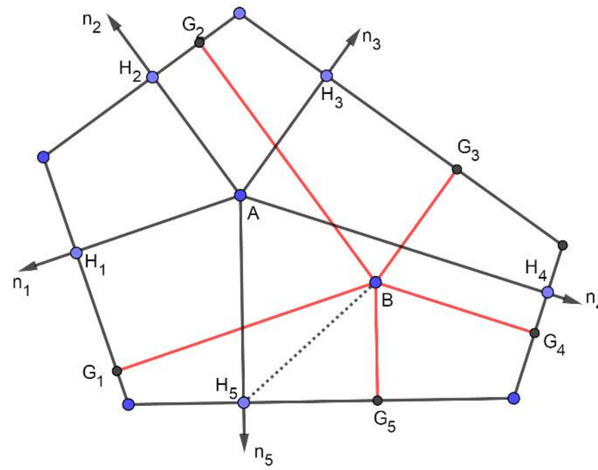
By Viviani's theorem for regular polygon, $V_{\mathcal{F}}$ is constant. Hence, $V_{\mathcal{H}}$ is constant. *(Q.E.D)*

However, the converse does not hold true, and a counterexample is the parallelogram, which has CVS property though its angles are not equal.

The proof is creative as it tries to reduce the problem to existed problem by construction of perpendicular and parallel lines. However, the drawback of it is that the construction may take time if done manually.

Beside Elias Abboud, another mathematician, Michel Cabart has attempted to prove this theorem using another way. With vector approach, his solution was without words, simply by constructing vectors of Hans Samelson (Literature Review 1)

Proof 2 (Michel Cabart)



Let A be a point inside the polygon, n_i unit vectors perpendicular to the ith side of the polygon, and H_i the feet of the perpendicular from A to the ith side. Since the polygon is equiangular, the angles between successive vectors n_i are equal, so that $\sum n_i = 0$.

The scalar product (AX, n_i), with X on the i th side, does not depend on the position of X.

The sum of distances from A to the sides of the polygon is:

$$S_A = \sum AH_i = \sum (AH_i, n_i).$$

For another point B with G_i being the feet of the perpendicular from B to the ith side of the polygon, the distance is:

$$S_B = \sum BG_i = \sum (BG_i, n_i) = \sum (BH_i, n_i).$$

So that:

$$S_A - S_B = (AB, \sum n_i) = 0.$$

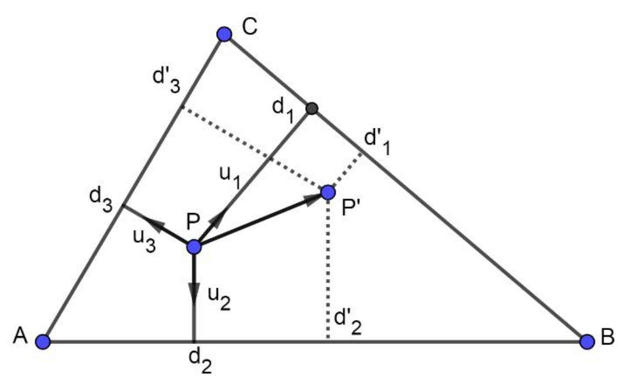
5. Converse of Viviani's theorem

The converse of Viviani's Theorem for equilateral triangles was proved to hold true (Zhibo Chen, Tian Liang, 2012). The converse theorem states:

> *Theorem 6.* *If, inside ABC, there is a circular region $\mathcal{R}$ for which the sum of the distances from a point P in R to the three sides of the triangle is independent of the position of P, then ABC is equilateral.*

The proof of Zhibo Chen and Tian Liang was greatly different from others since they approached the theorem in an unique way other than the usual approach which is using formula for area sum. Besides, this was also one of the earliest vector approach solution to Viviani's theorem and its related problems.

Proof



Let P be a point in $\mathcal{R}$, and let $\overrightarrow{u_1}, \overrightarrow{u_2}, \overrightarrow{u_3}$ be the unit vectors from P perpendicular to the sides of the triangle (see figure). Our goal is to show that the angle between any two of these vectors is 120◦, from which it will follow that each angle of the triangle is 60◦.

To this end, we first show that the sum of these vectors, $\vec{u} = \overrightarrow{u_1} + \overrightarrow{u_2} + \overrightarrow{u_3}$, is 0. Suppose not. From our hypothesis, it follows that there is a point P' in R so that , $\overrightarrow{PP'}$ is parallel to $\vec{u}$. Let $\vec{w}$ denote the vector $\overrightarrow{PP'}$, and let θ be the angle between $\overrightarrow{u_1}$ and $\vec{w}$. Further, let d_1, d_2, and d_3 be the distances from P to the sides of ABC, and let d'_1, d'_2, and d'_3 be the corresponding distances from P'. Note that by hypothesis $d_1 + d_2 + d_3 = d'_1 + d'_2 + d'_3$

On the one hand, $$\cos\theta = \frac{d_1 - d'_1}{|\vec{w}|},$$

while on the other hand, $$\cos\theta = \frac{\overrightarrow{u_1}.\vec{w}}{|\vec{w}|}$$ (since $\overrightarrow{u_1}$ is a unit vector).

Hence, $\overrightarrow{u_1} \,.\, \overrightarrow{w} = d_1 - d'_1$

and by symmetry $\overrightarrow{u_2} \,.\, \overrightarrow{w} = d_2 - d'_2$

$\overrightarrow{u_3} \,.\, \overrightarrow{w} = d_3 - d'_3$

It follows that $\vec{u} \cdot \vec{w} = 0$, and since these two vectors are parallel, it must be that $|\vec{u}| = 0$, a contradiction.

From this it follows that, for $i = 1, 2,$ and 3, $\overrightarrow{u_i} \cdot (\overrightarrow{u_1} + \overrightarrow{u_2} + \overrightarrow{u_3}) = 0.$

It is now straightforward to show that: $\overrightarrow{u_1} \,.\, \overrightarrow{u_2} = \overrightarrow{u_2} \,.\, \overrightarrow{u_3} = \overrightarrow{u_3} \,.\, \overrightarrow{u_1} = -\frac{1}{2}.$

Consequently, the angle between any pair of these vectors is $2\pi 3$, and so ABC must be equilateral.

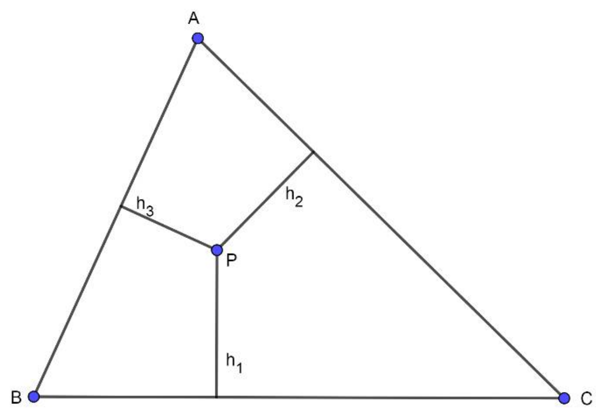
6. Conditions for polygons to possess CVS property

Conditions for polygons to possess CVS property (Elias Abboud, 2006), which was found and proven by linear programming, is stated as:

> *Theorem 7.* *If V takes equal values at three non-collinear points, inside a convex polygon, then the polygon has the CVS property.*

Proof

Given a triangle $\triangle ABC$, let a_1, a_2, a_3 be the lengths of the sides BC, AC, AB respectively. Let P be a point inside the triangle and let h_1, h_2, h_3 be the distances from the point P to the three sides respectively. Let V(P) be the sum of distances from point P to the sides of the triangle.



For $1 \le i \le 3$, let $x_i = \frac{h_i}{\sum_{i=1}^{3} h_i}$, where $\frac{h_i}{\sum_{i=1}^{3} h_i} = V(P)$

Clearly, for each $1 \le i \le 3$, we have $0 \le x_i \le 1$ and $\sum_{i=1}^{3} x_i = 1$. Denote $x = (x_1, x_2, x_3)$ and consider the linear function in three variables $F(x) = \sum_{i=1}^{3} a_i x_i$. Now, this function is closely related to the function V. Accurately,

$$F(x) = \sum_{i=1}^{3} a_i x_i = \frac{\sum_{i=1}^{3} a_i h_i}{\sum_{i=1}^{3} h_i} = \frac{2S}{V(P)}$$

where S is the area of the triangle.

Consequently, $F(x) = \sum_{i=1}^{3} a_i x_i$ takes equal values in a subset of points of the feasible region if and only if the function V takes equal values at the corresponding points inside the triangle.

Thus we may define the following linear programming problem; The objective function is:

$$F(x) = \sum_{i=1}^{3} a_i x_i$$

subject to the following constraints: $\begin{cases} \sum_{i=1}^{3} x_i \le 1 \\ x_i \ge 0\,; 1 \le i \le 3 \end{cases}$

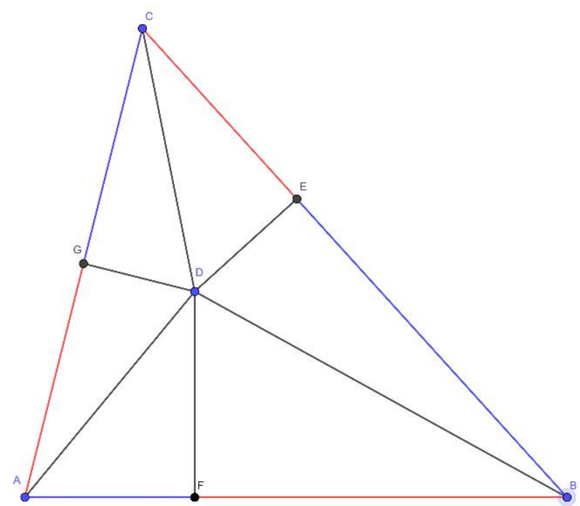
III. Carnot's Theorem

As we were doing research, we encountered this theorem that features a figure similar to that of Viviani's Theorem and it also deals with constant sums in a triangle. Therefore, we took an insight into the theorem and consider if we can somehow link it to Viviani's Theorem. Carnot's Theorem, established by Lazare Carnot, states that:

Theorem 7.1. For a triangle ABC, take interior point D from which we construct three perpendiculars to three sides. If E,F,G are points lying on those three perpendiculars on the sides, the following equation holds:

$$CG^2 + AF^2 + BE^2 = AG^2 + FB^2 + EC^2$$

Proof



By Pythagoras' Theorem:

$$CD^2 - CG^2 = GD^2$$
$$AD^2 - AG^2 = GD^2$$

$$CD^2 - CG^2 = AD^2 - AG^2$$

$$\Rightarrow \quad CD^2 - AD^2 = CG^2 - AG^2 \ (a)$$

Similarly, we have:

$$AD^2 - BD^2 = AF^2 - BF^2 \ (b)$$

$$BD^2 - CD^2 = BE^2 - CE^2 \ (c)$$

(a)+(b)+(c):

$$CD^2 - AD^2 + AD^2 - BD^2 + BD^2 - CD^2 = CG^2 - AG^2 + AF^2 - BF^2 + BE^2 - CE^2$$

$$CG^2 - AG^2 + AF^2 - BF^2 + BE^2 - CE^2 = 0$$

$$CG^2 + AF^2 + BE^2 = AG^2 + BF^2 + CE^2 \ (Q.E.D)$$

The converse of Carnot's theorem states that:

Theorem 7.2. For a triangle ABC, if E,F,G are the pedal points of three perpendiculars on the sides such that the following equation holds:

$$CG^2 + AF^2 + BE^2 = AG^2 + FB^2 + EC^2$$

Then the three perpendiculars must converge at a point.

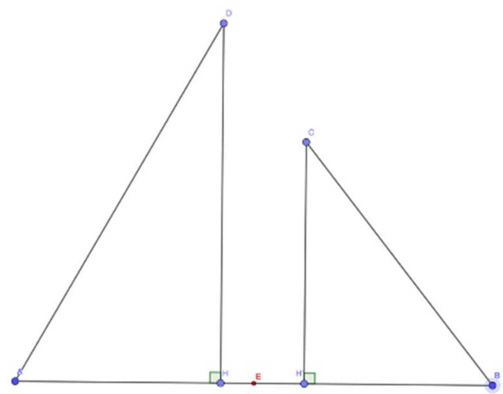
To prove the converse theorem, we would first prove two lemmas:

Lemma 1:

Given point A and B and a constant k. There exists one and only one point H on segment AB such that: **$HA^2 - HB^2 = k$**

Let E be the midpoint of AB.

$$HA^2 - HB^2 = k$$
$$\overline{HA}^2 - \overline{HB}^2 = k$$
$$(\overline{HA} - \overline{HB})(\overline{HA} + \overline{HB}) = k$$
$$\overline{BA}\,(\overline{HA} + \overline{HB}) = k$$
$$\overline{BA}\,(\overline{HE} + \overline{EA} + \overline{HE} + \overline{EB}) = k$$
$$\overline{BA}.\,2\overline{HE} = k$$
$$\therefore \overline{EH} = \frac{k}{2\,\overline{AB}}$$



k is constant, $\overline{AB}$ is constant, so $\overline{EH}$ is constant

∴ There exists only one and only one point H satisfying the equation since $\overline{EH}$ is a directed segment with given magnitude. *(Q.E.D)*

Lemma 2:

Given segment AB and line CD. CD is perpendicular to AB if and only if: **$CA^2 - CB^2 = DA^2 - DB^2$**

Let H and H' be the foot of the perpendicular from C and D to AB, respectively. It is given:

$$CA^2 - CB^2 = DA^2 - DB^2$$

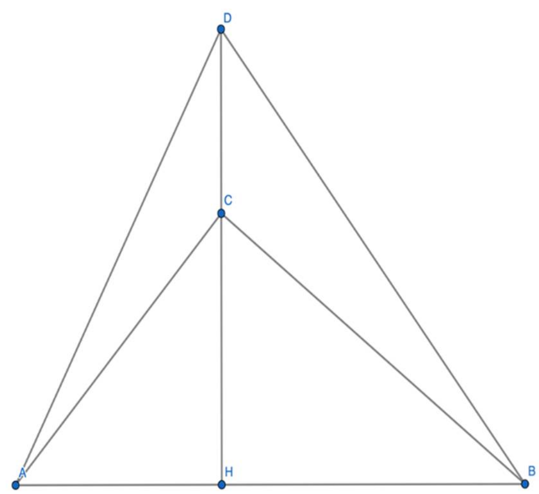
By Pythagoras' Theorem:

$$(HA^2 + HB^2) - (HB^2 + HC^2) = (H'A^2 + H'D^2) - (H'B^2 + H'D^2)$$

$$HA^2 - HB^2 = H'A^2 - H'B^2$$

By *Lemma 1:*

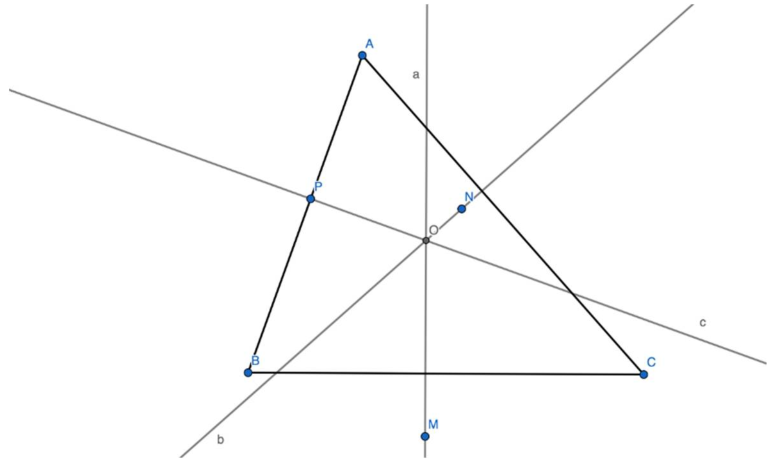
$$H \equiv H'$$
$$\Rightarrow CH \equiv DH'$$
$$\therefore CD \perp AB \text{ (Q.E.D)}$$



Proof of converse of Carnot's Theorem

By condition, *a* and *b* are respectively perpendicular to *BC* and *CA*. Since *BC*, *CA* intersect at *C*, *a* and *b* intersect.

Let *O* be the intersection of *a* and *b*. By **Lemma 2:**



a, b, c converge
$\Leftrightarrow O \in c$
$\Leftrightarrow PO \equiv c$
$\Leftrightarrow PO \perp AB$
$\Leftrightarrow PA^2 - PB^2 = OA^2 - OB^2$
$\Leftrightarrow (OB^2 - OA^2) + (PA^2 - PB^2) = 0$

$\Leftrightarrow (MB^2 - MC^2) + (NC^2 - NA^2) + (PA^2 - PB^2) = 0$ (Q.E.D)

INSPIRATION

From our research of existing work, we noted that the general method to approach Viviani's theorem and its extension is the application of the sum of area of triangles. So far, from the investigation on other studies that we have been referred to, there has been only 3 studies researching on Viviani's theorem using vectors. Those include: Zhibo Chen & Tiang Lian (2012), Hans Samelson (2003) and Michel Cabart. The study of Zhibo Chen & Tiang Lian (2012) is indeed to prove the converse of Viviani's theorem in triangles using the same method as Hans Samelson that working on unit vectors with trigonometry involved. The converse of Viviani's theorem in triangles indeed has been developed and extended to polygons, suggesting the conditions for a polygon to possess CVS property (Elias Abboud, 2009); however, the approach to the extension of the theorem in the broader context did not remain the same: linear programming was involved. For Secondary school students, additional work and study is required before a complete understanding on this way of approach. Therefore, we were inspired to follow closely the approach using vectors, which are believed to be more appropriate to give an insightful understanding to Secondary School students. With a powerful geometrical tool as vectors, it is confidently believed that gaps in the previous work would be clearly clarified and explained.

Besides, there have been attempts to further investigate problems related to Viviani's theorem such as loci of points, deduction of eclipse, Miquel triangles, etc. However, it has been observed that there is no attempt to establish a link between Viviani's theorem with another geometrical theorem in the same field. Therefore, to show theorems in Mathematics are thoroughly coherent, we decided to introduce a relationship between theorems in the same Geometry field.

METHODOLOGY

At the very start, when sourcing for a field of Mathematics to investigate, we looked into previous year's published submissions to get a rough idea of how extended our projects should be. Among different areas of Math, we found Geometry to be the least complicated to approach and extend as most of the theorems and problems can be visualised with figures and they hardly have fixed solutions.

Next, we searched the Internet, specifically sites such as *Wikipedia.com, cut-the-knot.org, xmltwo.ibo.org*, for papers, reports and articles providing deeper insights into some problems. After putting all problems and theorems we felt interested in into a spreadsheet, we compared them using common factors like:

availability of previous work, applicability, etc. We narrowed our choices down to theorems related to triangles, because triangles are the most basic polygons (3-sided polygons), which promises further extension by altering conditions, such as increasing the number of sides or introducing new lines.

Finally, we decided that Viviani's Theorem would be the topic of our project on the grounds that it features basic constructions that are easy to follow, that its algebraic components can be recorded and observed using basic programming on software like Microsoft Excel, and that it inspires us with different ways to extend.

During our project, we used the online application, GeoGebra to confirm the conditions and requirements needed for our extension and proof of the Viviani's Theorem, as well as to create shapes used in our report and presentation. We constructed the figure based on the original conditions of the theorem, then tried dragging points around and adding/eliminating points and lines. To examine the CVS property of each polygon, we wrote a sum function of all the distances and linked it to an Excel sheet, which automatically recorded the sums as we changed the position of the arbitrary interior point. This helps us a lot in testing our hypothesis and spotting the exceptions.

RESULTS

I. VIVIANI'S THEOREM AND ITS EXTENSION

1. Vectors approach

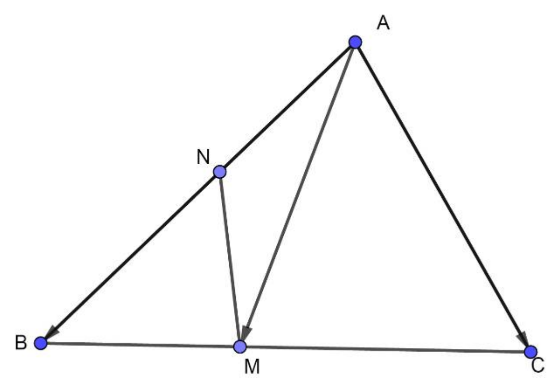
Theorem 1.1 All regular polygons possess CVS property

Lemma 1:

M is a point on side *BC* of triangle *ABC*. We have the equation:

$$\overrightarrow{AM} = \frac{MC}{BC}\overrightarrow{AB} + \frac{MC}{BC}\overrightarrow{AC}$$

Let *N* be a point on *AB* such that *MN // AC*. (1.1)



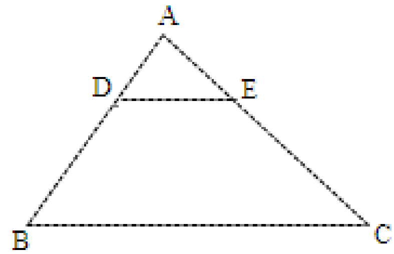
By Thales' Theorem:

$\frac{AN}{AB} = \frac{MC}{BC}$; $\frac{NM}{AC} = \frac{MB}{BC}$ (1.2)

From (1.1) and (1.2):

$\overrightarrow{AM} = \frac{MC}{BC}\overrightarrow{AB} + \frac{MC}{BC}\overrightarrow{AC}$ *(Q.E.D)*

***__Thales' Theorem__**: Given triangle ABC, D and E are on AB and AC respectively such that DE is parallel to BC. The following result holds:*



$$\text{According to Thales theorem: } \frac{AD}{BD} = \frac{AE}{CE}$$

Lemma 2:

> The inscribed circle of triangle *ABC* touches each side *BC, CA, AB* at *D, E, F*, respectively. Denote length of *BC, AC, AB* as *a, b, c*. We have the equation:
>
> $$\mathbf{a}\overrightarrow{ID} + \mathbf{a}\overrightarrow{IE} + \mathbf{a}\overrightarrow{IF} = 0$$

Let *A'* be the point of intersection of *AI* and *BC*. By characteristic of the bisector:

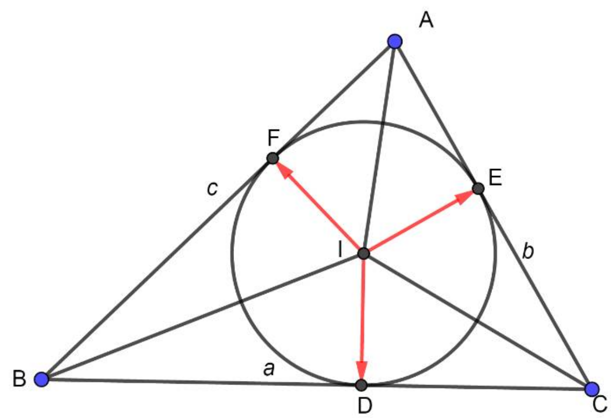
$$\frac{A'C}{A'B} = \frac{b}{c}$$

$$\Rightarrow \frac{A'C}{b} = \frac{A'B}{c} = \frac{A'B + A'C}{b+c} = \frac{a}{b+c} \quad (2.1)$$

And $\frac{IA'}{IA} = \frac{BA'}{BA} = \frac{\frac{ac}{b+c}}{c} = \frac{a}{b+c}$ (2.2)

In triangle *IBC*, from ***Lemma 1***:

$$\overrightarrow{IA'} = \frac{A'C}{BC}\overrightarrow{IB} + \frac{A'B}{BC}\overrightarrow{IC} \quad (2.3)$$



From (2.1) and (2.3): $\overrightarrow{IA'} = \frac{b}{b+c}\overrightarrow{IB} + \frac{c}{b+c}\overrightarrow{IC}$ (2.4)

Also, $\overrightarrow{IA'} = -\frac{IA'}{IA}\overrightarrow{IA}$ (2.5)

From (2.2) and (2.5): $\overrightarrow{IA'} = -\frac{a}{b+c}\overrightarrow{IA}$ (2.6)

From (2.4) and (2.6): $a\overrightarrow{IA} + b\overrightarrow{IB} + c\overrightarrow{IC} = 0$ (*)

Let $AE = AF = x$; $BF = BD = y$; $CD = CE = z$

$$y + z = a, z + x = b, x + y = c$$

In triangle *IBC, ICA, IAB*, from *Lemma 1*:

$$\begin{cases} a\overrightarrow{ID} = z\overrightarrow{IB} + y\overrightarrow{IC} \\ b\overrightarrow{IE} = x\overrightarrow{IC} + z\overrightarrow{IA} \\ c\overrightarrow{IF} = y\overrightarrow{IA} + x\overrightarrow{IB} \end{cases}$$

⇨ $a\overrightarrow{ID} + a\overrightarrow{IE} + a\overrightarrow{IF} = (x+y)\overrightarrow{IA} + (z+x)\overrightarrow{IB} + (x+y)\overrightarrow{IC}$

⇨ $a\overrightarrow{ID} + a\overrightarrow{IE} + a\overrightarrow{IF} = a\overrightarrow{IA} + b\overrightarrow{IB} + c\overrightarrow{IC}$ (**)

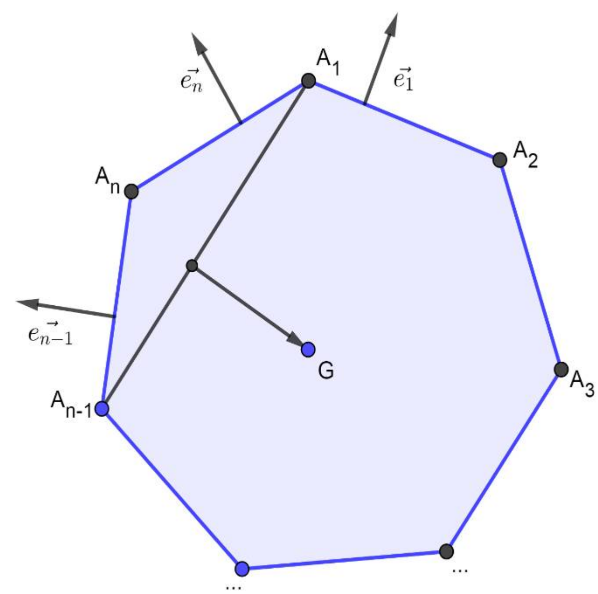
From (*) and (**):

$$a\overrightarrow{ID} + a\overrightarrow{IE} + a\overrightarrow{IF} = 0 \text{ (Q.E.D)}$$

Lemma 3.

> The sum of unit vectors perpendicular to the sides of a regular polygon is a zero vector.

Given a convex polygon $A_1A_{2....}A_n$ with unit vectors $\vec{e_i}$



($1 \leq i \leq n$) directing outwards of the polygon and are perpendicular to $A_i\vec{A}_{i+1}$ (considering $A_{n+1} \equiv A_1$) respectively.

We are going to prove that: $A_1A_2\vec{e}_1 + A_2A_3\vec{e}_2 + \cdots + A_nA_1\vec{e}_n = \vec{0}$

For *n=3*, the polygon is a triangle. From *lemma 2*, the result holds true for *n=3* (3.1)

Supposing the theorem is true for all $(n-1)$-sided polygon *(n > 4)* (3.2)

Construct a triangle $A_1A_{n-1}A_n$ outside the $(n-1)$-sided polygon such that the new *n*-sided polygon is a convex polygon.

Then, construct unit vector $\overrightarrow{e_\iota}$ perpendicular to A_iA_{i+1}, directing outwards of triangle $A_1A_{n-1}A_n$.

Apply (3.1) and (3.2) correspondingly to triangle $A_1A_{n-1}A_n$ and $(n-1)$-sided polygon:

$$A_{n-1}A_n\vec{e}_{n-1} + A_1A_{n-1}(-\vec{e}) + A_1A_n\overrightarrow{e_n} = \vec{0} \text{ (3.3)}$$

$$A_1A_2\vec{e}_1 + A_2A_3\vec{e}_2 + \cdots + A_{n-1}A_1\overrightarrow{e_n} = \vec{0} \text{ (3.4)}$$

Take (3.3) + (3.4), it holds: $A_1A_2\vec{e}_1 + A_2A_3\vec{e}_2 + \cdots + A_nA_1\vec{e}_n = \vec{0}$ (3.5)

Therefore, using induction, the theorem applies for *n*-sided convex polygons.

When the polygon is regular:

$$A_1A_2 = A_2A_3 = A_3A_4 = \ldots = A_nA_1 = k$$

(*k* is the side length of the polygon)

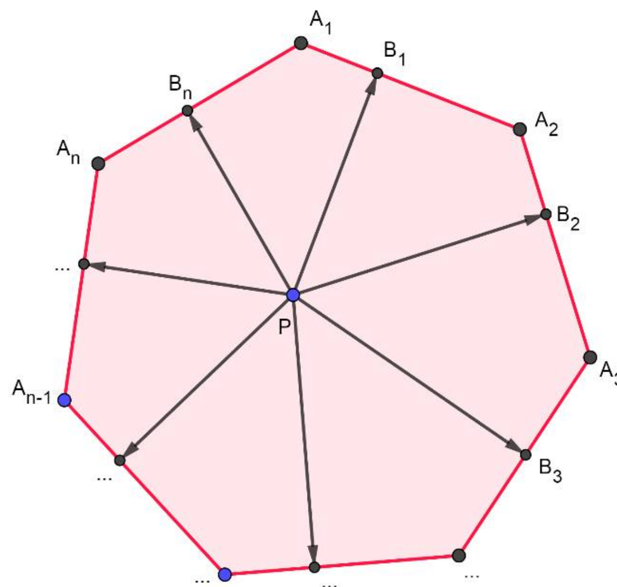
From (3.5):
$$A_1A_2\vec{e}_1 + A_2A_3\vec{e}_2 + \cdots + A_nA_1\vec{e}_n = \vec{0}$$
$$k.\vec{e}_1 + k.\vec{e}_2 + \cdots + k.\vec{e}_n = \vec{0}$$
$$k.\sum_{i=1}^{n}\vec{e_i} = 0$$
$$\therefore \sum_{i=1}^{n}\vec{e_i} = 0$$

Proof of Theorem 1.

Construct unit vectors perpendicular and directed to the sides of given polygon from interior point *P*.

Let *V: R* → *R* be the function of the sum of distance from an interior point to sides of the polygon. Any polygon that has constant sum of distance *V* is said to have CVS property. It holds:

$$\cos\angle A_1PB_1 = \frac{PB_1}{PA_1}$$
$$PB_1 = PA_1.\frac{PB_1}{PA_1}$$
$$= PA_1.\,1\,.\cos\angle A_1PB_1$$
$$= \overrightarrow{PA_1}.\,\vec{e}_1$$
$$\therefore PB_1 = \overrightarrow{PA_1}.\vec{e}_1$$



Similarly:
$$PB_2 = \overrightarrow{PA_2}.\,\vec{e}_2$$
$$PB_3 = \overrightarrow{PA_3}.\,\vec{e}_3$$
$$\cdots$$
$$PB_n = \overrightarrow{PA_n}.\,\vec{e}_n$$

The sum of the unit vectors from interior point *P* that are perpendicularly directed to all the sides of the polygon is $V(P) = \sum_{i=1}^{n}\overrightarrow{PA_i}\cdot\vec{e}_i$

Choosing another interior point *Q*, similarly, the sum of distances from interior point *Q* to sides of the polygon is $V(Q) = \sum_{i=1}^{n}\overrightarrow{QA_i}\cdot\vec{e}_i$

The polygon possesses CVS property if the sum of distances of any given interior point of the polygon is a constant. Therefore, in order to prove that the regular polygon possesses CVS property, we have to prove *V(P)= V(Q).*

$$V(P) - V(Q) = \sum_{i=1}^{n}\overrightarrow{PA_i}\cdot\vec{e}_i - \sum_{i=1}^{n}\overrightarrow{QA_i}\cdot\vec{e}_i$$
$$= \sum_{i=1}^{n}\vec{e}_i(\overrightarrow{PA_i} - \overrightarrow{QA_i}) = \overrightarrow{PQ}\cdot(\sum_{i=1}^{n}\vec{e_i}) = \vec{0}\ (*)$$

From **Lemma 3**, the sum of unit vectors perpendicular to the sides of a regular polygon is a zero vector. Hence, the above result is a zero vector.

From here we can conclude that for any two random points *P* and *Q* chosen in the given polygon, $V(P) = V(Q)$

$\therefore$ Thus, the regular polygon possesses CVS property.

> Theorem 1.2 If the sum of unit vectors perpendicular to the sides of the polygon is a zero vector, the polygon possesses CVS property.

Similarly, to the proof of Theorem 1.1, let P and Q be two arbitrary points in the given polygon.

It holds: $$\begin{cases} V(P) = \sum_{i=1}^{n} \overrightarrow{PA_i} \cdot \vec{e}_i \\ V(Q) = \sum_{i=1}^{n} \overrightarrow{QA_i} \cdot \vec{e}_i \end{cases}$$

$V(P) - V(Q) = \sum_{i=1}^{n} \overrightarrow{PA_i} \cdot \vec{e}_i - \sum_{i=1}^{n} \overrightarrow{QA_i} \cdot \vec{e}_i = \sum_{i=1}^{n} \vec{e}_i (\overrightarrow{PA_i} - \overrightarrow{QA_i}) = \overrightarrow{PQ} \cdot (\sum_{i=1}^{n} \vec{e_i}) = \vec{0}$ (*)

Since $\sum_{i=1}^{n} \vec{e_i}) = \vec{0}$, (*) holds true: $V(P) = V(Q)$. Thus, the polygon possesses CVS property.

Similarly, P and Q are interior points of the given polygon. From the assumption:

> Theorem 1.3 If a polygon possesses CVS property, the sum of unit vectors perpendicular to the sides of the polygon is a zero vector.

$$V(P) = V(Q) = k \text{ (a constant)}$$

$$V(P) = \sum_{i=1}^{n} \overrightarrow{PA_i} \cdot \vec{e}_i$$

$$V(Q) = \sum_{i=1}^{n} \overrightarrow{QA_i} \cdot \vec{e}_i$$

$$V(P) - V(Q) = \vec{0}$$

$$V(P) - V(Q) = \sum_{i=1}^{n} \overrightarrow{PA_i} \cdot \vec{e}_i - \sum_{i=1}^{n} \overrightarrow{QA_i} \cdot \vec{e}_i = \overrightarrow{PQ} \cdot (\sum_{i=1}^{n} \vec{e_i}) = \vec{0} \ (*)$$

Since P and Q are two distinct points, $\overrightarrow{PQ}$ is not a zero vector.

However, from here, we cannot conclude that the sum of unit vectors that are perpendicular to the sides of the polygon is a zero vector even though vector PQ is a non zero vector. Noted that this conclusion is the key condition for a convex polygon to have CVS property.

In fact, from (*), the deduction should be:

$$\sum_{i=1}^{n} \vec{e_i} = \vec{0} \ (4.1)$$

Or $$\sum_{i=1}^{n} \vec{e_i} = \vec{e} \text{ and } \vec{e} \perp \overrightarrow{PQ} \ (4.2)$$

When $\overrightarrow{PQ}$ is perpendicular to the sum-vector. This results in $\overrightarrow{PQ} \cdot \sum_{i=1}^{n} \vec{e} = 0$, regardless of whether or not the polygon has CVS property.

Choosing another arbitrary point Z inside the given polygon. Since the polygon possesses CVS property:

$$V(P)=V(Q)=V(Z)$$

$$V(P) - V(Q) = 0$$

$$\sum_{i=1}^{n} \overrightarrow{PA_i} \cdot \vec{e_i} - \sum_{i=1}^{n} \overrightarrow{QA_i} \cdot \vec{e_i} = \sum_{i=1}^{n} \overrightarrow{PQ} \cdot \vec{e_i} = \overrightarrow{PQ} \cdot (\sum_{i=1}^{n} \vec{e_i}) = \vec{0} \qquad (4.3)$$

$$V(Q) - V(Z) = 0$$

$$\sum_{i=1}^{n} \overrightarrow{QA_i} \cdot \vec{e_i} - \sum_{i=1}^{n} \overrightarrow{ZA_i} \cdot \vec{e_i} = \sum_{i=1}^{n} \overrightarrow{QZ} \cdot \vec{e_i} = \overrightarrow{QZ} \cdot \left(\sum_{i=1}^{n} \vec{e_i}\right) = \vec{0} \qquad (4.4)$$

If *P, Q* and *Z* are collinear, the result (4.3) obviously leads to (4.4) without the polygon possessing CVS property. In this case, choosing another point *Z* simply does not make any more significant deduction than using two points *P* and *Q*.
If *P, Q* and *Z* are non-collinear, we can conclude that the sum of unit vectors is a zero vector. If the sum of unit vector is a non-zero vector, $\overrightarrow{PQ}$ and $\overrightarrow{QZ}$ must be both perpendicular to this sum-vector. Therefore, *P, Q* and *Z* are on the same line (which is contradictory to our assumption).
Thus, using 3 non-collinear points, our hypothesis on the converse of Theorem 1.2 has been proved.

<u>Theorem 1.3:</u> What can we tell about the geometrical property of polygons that possess CVS property?

Given a convex polygon. Construct unit vectors from an interior point perpendicularly directed to the sides of the polygon. If $\sum_{i=1}^{n} \vec{e_i} = \vec{0}$, then the polygon has CVS property.

Cases for an *n*-sided polygon to have $\sum_{i=1}^{n} \vec{e_i} = \vec{0}$,

- If the number of sides of polygon is odd,
  - The polygon is regular
- If the number of sides of polygon is even,
  - The polygon is regular
  - There must be $\frac{n}{2}$ pairs of vectors that are in opposite direction

➔ The polygon has opposite sides that are parallel, since the opposite sides share the same perpendicular from the given interior point

2. Corollaries

<u>Corollary 1</u>. Given a regular *n*-sided polygon with each side having a length of *a*. Let $B_1, B_2, B_3 \dots B_n$ be the foot of the perpendicular lines from a random interior point *P* to the sides of the polygon, which we defined as $A_1A_2, A_2A_3, \dots A_nA_1$. *O* is another random interior point. The following equation holds:

$$\sum_{i=1}^{n} \overrightarrow{A_iA_{i+1}} \cdot \overrightarrow{PB_{i+1}} = \sum_{i=1}^{n} \overrightarrow{A_iA_{i+1}} \cdot \overrightarrow{OB_{i+1}} = 2\, area\ of\ polygon \times \cos\left(90 - \frac{360}{n}\right)^\circ$$

Let $\overrightarrow{A_iA_{i+1}} \cdot \overrightarrow{PB_{i+1}} = \overrightarrow{A_iA_{i+1}} \cdot (\overrightarrow{PO} + \overrightarrow{OB_{i+1}})$

From the equation above,

$$\overrightarrow{A_1A_2} \cdot \overrightarrow{PB_2} = A_1A_2 \cdot PB_2 \cdot \cos\left(90 - \frac{360}{n}\right)^\circ$$

$$\overrightarrow{A_2A_3} \cdot \overrightarrow{PB_3} = A_2A_3 \cdot PB_3 \cdot \cos\left(90 - \frac{360}{n}\right)^\circ$$

…

…

$$\overrightarrow{A_nA_1} \cdot \overrightarrow{PB_n} = A_nA_1 \cdot PB_1 \cdot \cos(90 - \frac{360}{n})°$$

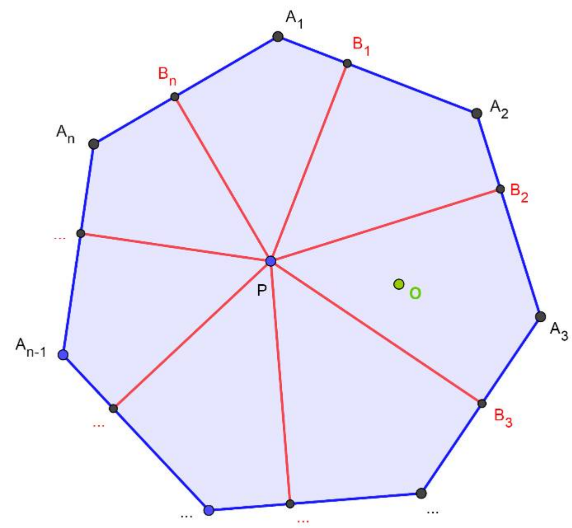
Hence,

$$\sum_{i=1}^{n} \overrightarrow{A_iA_{i+1}} \cdot \overrightarrow{PB_{i+1}} = a \cdot \sum_{i=1}^{n} PB_i \cdot \cos(90 - \frac{360}{n})°$$

On the other hand:

$$\sum_{i=1}^{n} PB_i = 2\,\frac{area\ of\ polygon}{a}$$

Thus, Corollary 1 is true.



<u>Corollary 2</u>. (hypothesis) Given an equilateral polygon. *P* and *O* are arbitrary points inside the given polygon. From *P*, construct perpendicular lines to sides of polygon. Let $B_1, B_2, B_3, \ldots$ be the intersections of the constructed perpendicular lines to sides $A_1A_2, A_2A_3, A_3A_4, \ldots$ The given polygon is regular if:

$$\sum_{i=1}^{n} \overrightarrow{A_iA_{i+1}} \cdot \overrightarrow{PB_{i+1}} = \sum_{i=1}^{n} \overrightarrow{A_iA_{i+1}} \cdot \overrightarrow{OB_{i+1}} = 2\ area\ of\ polygon \times \cos(90 - \frac{360}{n})°$$

Given a *n*-sided convex polygon, it holds:

$$\overrightarrow{A_1A_2} \cdot \overrightarrow{PB_2} = A_1A_2 \cdot PB_2 \cdot \cos(\angle A_1A_2A_3 - 90)°$$

$$\overrightarrow{A_2A_3} \cdot \overrightarrow{PB_3} = A_2A_3 \cdot PB_3 \cdot \cos(\angle A_2A_3A_4 - 90)°$$

…

$$\overrightarrow{A_nA_1} \cdot \overrightarrow{PB_n} = A_nA_1 \cdot PB_1 \cdot \cos(\angle A_nA_1A_2 - 90)°$$

$$\sum_{i=1}^{n} \overrightarrow{A_iA_{i+1}}.\overrightarrow{PB_{i+1}} = a.\sum_{i=1}^{n} PB_i.\cos(\angle A_iA_{i+1}A_{i+2} - 90)°$$

$$= 2.\,area\ of\ the\ polygon.\cos\,(90 - \frac{360}{n})$$

$$Area\ of\ equilateral\ convex\ polygon = \frac{1}{2}a.\sum_{i=1}^{n} PB_i$$

$$\sum_{i=1}^{n} \overrightarrow{A_iA_{i+1}}.\overrightarrow{PB_{i+1}} = a.\sum_{i=1}^{n} PB_i.\cos(\angle A_iA_{i+1}A_{i+2} - 90)° = a.\sum_{i=1}^{n} PB_i\ .\cos\,(90 - \frac{360}{n})$$

$$\sum_{i=1}^{n} PB_i.\cos(\angle A_iA_{i+1}A_{i+2} - 90)° = \sum_{i=1}^{n} PB_i\ .\cos\,(90 - \frac{360}{n})\ (*)$$

In order to prove that the polygon is regular (both equilateral and equiangular), from (*) we have to prove that: $\angle A_1A_2A_3 = \angle A_2A_3A_4 = \ldots = \angle A_nA_1A_2$

Unsolved challenge

Given that $\theta_1, \theta_2, \ldots \theta_n$ are interior angles of an $n-$ sided polygon such that:

$$\sum_{i=1}^{n} \boldsymbol{PB_i} \cdot \boldsymbol{cos(\theta_i - 90)^\circ} = \left(\sum_{i=1}^{n} \boldsymbol{PB_i}\right) \cdot \boldsymbol{cos\left(90 - \frac{360}{n}\right)^\circ}$$

And $\sum_{i=1}^{n} \boldsymbol{\theta_i} = \boldsymbol{180(n-2)}$

Prove that: $\boldsymbol{\theta_1} = \boldsymbol{\theta_2} = \boldsymbol{\theta_3} = \cdots = \boldsymbol{\theta_n}$

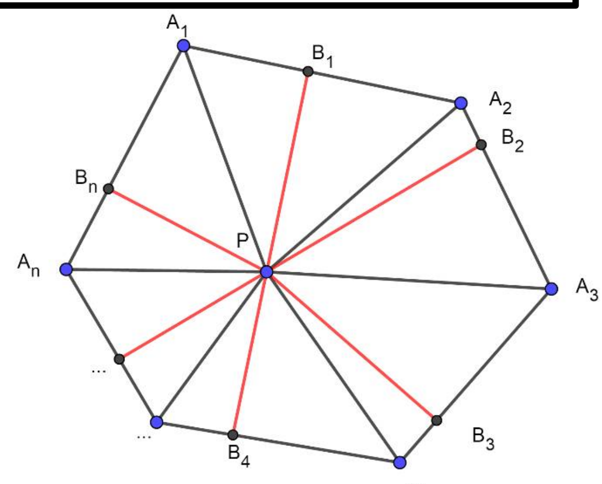
II. Carnot's Theorem and its Relationship to Viviani's Theorem

1. Extended Carnot's Theorem and its Converse

Theorem 2.1 Given an n-sided polygon $A_1A_2A_3\ldots A_n$. P is an interior point of the given polygon. $B_1, B_2, B_3,.., B_n$ lie on $A_1A_2, A_2A_3, A_3A_4,..., A_nA_1$ respectively. The following holds true:

$$\sum_{i=1}^{n} A_iB_i^{\ 2} = \sum_{i=1}^{n} B_iA_{i+1}^{\ 2}$$

By Pythagoras' Theorem:



$$PA_n^{\ 2} - A_nB_n^{\ 2} = PB_n^{\ 2}$$

$$PA_1^{\ 2} - B_nA_1^{\ 2} = PB_n^{\ 2}$$

⇨ $PA_n^{\ 2} - A_nB_n^{\ 2} = PA_1^{\ 2} - B_nA_1^{\ 2}$

⇨ $PA_n^{\ 2} - PA_1^{\ 2} = A_nB_n^{\ 2} - B_nA_1^{\ 2}$ (5.1)

Similarly,

$$PA_1^{\ 2} - PA_2^{\ 2} = A_1B_1^{\ 2} - B_1A_2^{\ 2} \quad (5.2)$$

$$\ldots$$

$$PA_{n-1}^{\ 2} - PA_n^{\ 2} = A_{n-1}B_{n-1}^{\ 2} - B_{n-1}A_n^{\ 2} \quad (5.n)$$

Sum of *(5.1) + (5.2) + … + (5.n):*

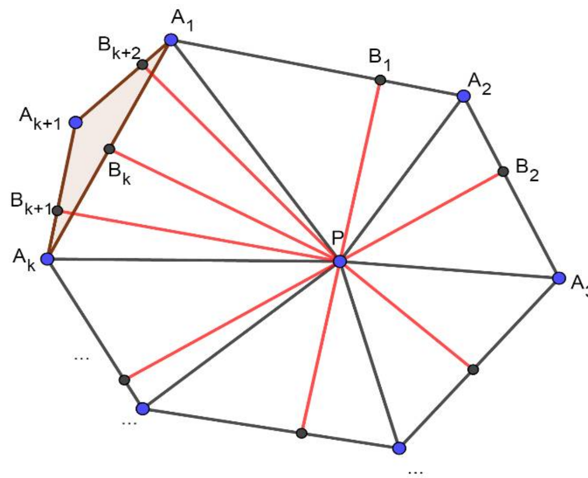
$$0 = A_nB_n^{\ 2} - B_nA_1^{\ 2} + A_1B_1^{\ 2} - B_1A_2^{\ 2} + \ldots + A_{n-1}B_{n-1}^{\ 2} - B_{n-1}A_n^{\ 2}$$

$$A_1B_1^{\ 2} + \ldots + A_{n-1}B_{n-1}^{\ 2} + A_nB_n^{\ 2} = B_1A_2^{\ 2} + \ldots + B_{n-1}A_n^{\ 2} + B_nA_1^{\ 2}$$

$$\therefore \sum_{i=1}^{n} A_iB_i^{\ 2} = \sum_{i=1}^{n} B_iA_{i+1}^{\ 2} \ (Q.E.D)$$

<u>Theorem 2.2</u> In an *n*-sided polygon, take one point $B_1, B_2, \ldots, B_n$ on each side A_1A_2, A_2A_3,..., A_nA_1. The lines that pass through points B_i and perpendicular to side A_iA_{i+1} will converge at one point if :

$$\sum_{i=1}^{n} \boldsymbol{A_iB_i}^2 = \sum_{i=1}^{n} \boldsymbol{B_iA_{i+1}}^2$$



As proven in *Lit.rev 7.*, the converse of extended Carnot's Theorem holds true for triangles when *n=3.* (6.1)

Assume that the converse of extended Carnot's Theorem also holds true for k-sided polygons:

$A_1B_1{}^2 + \ldots + A_{k-1}B_{k-1}{}^2 + A_kB_k{}^2$

$= B_1A_2{}^2 + \ldots + B_{k-1}A_k{}^2 + B_kA_1{}^2$(6.2)

Construct a triangle $A_1A_kA_{k+1}$outside the k-sided polygon such that the newly constructed (k+1)-sided polygon is a convex polygon. (6.3) Now we need to prove that the theorem also holds true for this newly constructed (k+1)-sided polygon.

Thus we have to prove the following equation:

$A_1B_1{}^2 + \ldots + A_kB_{k+1}{}^2 + A_{k+1}B_{k+2}{}^2 = B_1A_2{}^2 + \ldots + B_{k+1}A_{k+1}{}^2 + B_{k+2}A_1{}^2$ (*)

From (6.1) in triangle $A_kA_{k+1}A_1$:

$$A_kB_{k+1}{}^2 + A_{k+1}B_{k+2}{}^2 + A_1B_k{}^2 = B_{k+1}A_{k+1}{}^2 + B_{k+2}A_1{}^2 + B_kA_k{}^2 \text{(6.4)}$$

Take *(2)*- *(4)*:

$(A_1B_1{}^2 + \ldots + A_{k-1}B_{k-1}{}^2 + A_kB_k{}^2) - (B_{k+1}A_{k+1}{}^2 + B_{k+2}A_1{}^2 + B_kA_k{}^2)$

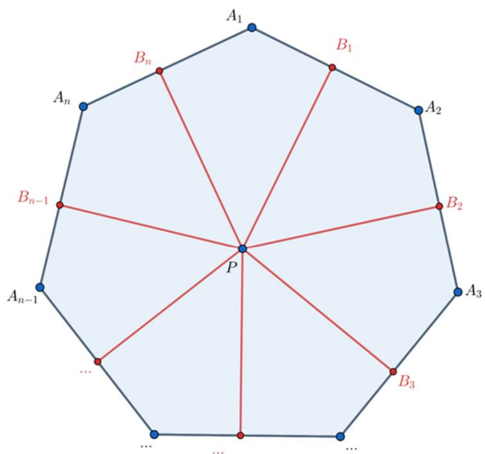
$= (B_1A_2{}^2 + \ldots + B_{k-1}A_k{}^2 + B_kA_1{}^2)$ - $(A_kB_{k+1}{}^2 + A_{k+1}B_{k+2}{}^2 + A_1B_k{}^2)$

$\Leftrightarrow A_1B_1{}^2 + \ldots + A_{k-1}B_{k-1}{}^2 + A_kB_k{}^2 - B_{k+1}A_{k+1}{}^2 - B_{k+2}A_1{}^2 - B_kA_k{}^2 = B_1A_2{}^2 + \ldots + B_{k-1}A_k{}^2 + B_kA_1{}^2 - A_kB_{k+1}{}^2 - A_{k+1}B_{k+2}{}^2 - A_1B_k{}^2$

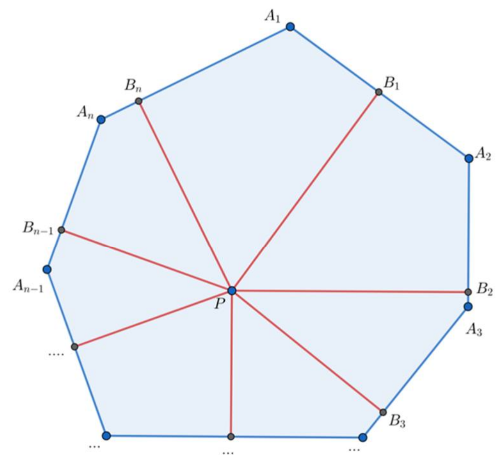
$\Leftrightarrow A_1B_1{}^2 + \ldots + A_{k-1}B_{k-1}{}^2 - B_{k+1}A_{k+1}{}^2 - B_{k+2}A_1{}^2 = B_1A_2{}^2 + \ldots + B_{k-1}A_k{}^2 - A_kB_{k+1}{}^2 - A_{k+1}B_{k+2}{}^2$

$\Leftrightarrow A_1B_1{}^2 + \ldots + A_{k-1}B_{k-1}{}^2 + A_kB_{k+1}{}^2 + A_{k+1}B_{k+2}{}^2 = B_1A_2{}^2 + \ldots + B_{k-1}A_k{}^2 + B_{k+1}A_{k+1}{}^2 + B_{k+2}A_1{}^2$ *(Q.E.D)*

2. The link between extended Carnot's Theorem and extended Viviani'



EXTENDED VIVIANI'S THEOREM



EXTENDED CARNOT'S THEOREM

We observed some similarities in the model of the two theorems, so we try to establish a link between them:

<u>**Suggested solution for Linked problem 1:**</u>

<u>Linked problem 1</u> Given an n-sided regular polygon $A_1A_2A_3\ldots A_n$ and a system of point $B_1, B_2, B_3, \ldots, B_n$ on the sides $A_1A_2, A_2A_3, A_3A_4, \ldots, A_nA_1$ respectively such $that$ $\sum_{i=1}^{n} \boldsymbol{A_iB_i}^2 = \sum_{i=1}^{n} \boldsymbol{A_{i+1}B_i}^2$

O is an interior point of the polygon.

Prove that: $\sum_{i=1}^{n} \overrightarrow{\boldsymbol{A_\iota A_{\iota+1}}}.\overrightarrow{\boldsymbol{OB_{\iota+1}}}$ is independent from the location of system of points Bi and the position of point O.

First, we are going to prove that the lines that pass through the $B_1, B_2, B_3, \ldots, B_n$ and perpendicular to the respective sides $A_1A_2, A_2A_3, A_3A_4, \ldots, A_nA_1$ converge at one point. This is actually the **extension of Carnot's theorem in polygons**.

Let P be the point of converging of those lines.

Since the polygon is regular, the polygon will possess CVS property.

Using **Corollary 1,** it holds:

$$\sum_{i=1}^{n} \overrightarrow{\boldsymbol{A_\iota A_{\iota+1}}}.\overrightarrow{\boldsymbol{PB_{\iota+1}}} = \sum_{i=1}^{n} \overrightarrow{\boldsymbol{A_\iota A_{\iota+1}}}.\overrightarrow{\boldsymbol{OB_{\iota+1}}} = \boldsymbol{2\ area\ of\ the\ polygon} \times \boldsymbol{cos}(\boldsymbol{90°} - \frac{\boldsymbol{360°}}{\boldsymbol{n}})$$

As $\sum_{i=1}^{n} \overrightarrow{A_\iota A_{\iota+1}}.\overrightarrow{OB_{\iota+1}} = 2\ area\ of\ the\ polygon \times cos(90° - \frac{360°}{n})$,

⇨ The sum is a constant.

∴ Therefore, it is independent from the location of the system of points B_i and the location of point O. *(Q.E.D)*

<u>Linked problem 2</u>. Given an equilateral n-sided polygon $A_1A_2A_3...A_n$.

A system of points $B_1, B_2, B_3, ..., B_n$ are points on lying on the sides of the polygon $A_1A_2, A_2A_3, A_3A_4, ..., A_nA_1$ respectively. O is an interior point in the given polygon.

Prove that the polygon has CVS property if and only if:

$$\begin{cases} \sum_{i=1}^{n} A_iB_i^{\,2} = \sum_{i=1}^{n} A_{i+1}B_i^{\,2} \\ \sum_{i=1}^{n} \overrightarrow{A_iA_{i+1}}.\overrightarrow{OB_{i+1}} = 2\, area\ of\ polygon \times cos(90° - \frac{360°}{n}) \end{cases}$$

<u>Suggested solution for Linked problem 2.</u>

First, since $\sum_{i=1}^{n} A_iB_i^{\,2} = \sum_{i=1}^{n} A_{i+1}B_i^{\,2}$, Lines that pass through the $B_1, B_2, B_3, …, B_n$ and perpendicular to the respective sides $A_1A_2, A_2A_3, A_3A_4, …, A_nA_1$ converge at one point.

In fact, this is the extension of Carnot's theorem in polygons.

Let P be the point of convergence of those lines.

$$\sum_{i=1}^{n} \overrightarrow{A_iA_{i+1}}.\overrightarrow{OB_{i+1}} = 2\, area\ of\ polygon \times \cos(90° - \frac{360°}{n})$$

Using the result of **Corollary 2**, the given equilateral polygon is regular

⇨ All regular polygon possesses CVS property. *(Q.E.D)*

DISCUSSION

I. Viviani's theorem and its extension

Vectors approach

So far, area formula is the most widely used method and applied in most of the studies of the extensions of Viviani's theorem. However, as the project is to further study Viviani's theorem and its extension using Vectors, the entire process is merely based on the application of vectors in extending Viviani's theorem and studying some of its related problems. The first approach to the extended Viviani's theorem is to prove Theorem 1.1, one of the most significant results and most basic extensions extending the original theorem from equilateral triangles to regular polygons.

Three lemmas have been proved and used to support the proof in vectors. In fact, there is an incorporated connection between the three lemmas in proving Extension 1. The result of Lemma 1 is used to yield the result of Lemma 2. When generalizing Lemma 2 from the specific case of $n=3$ to $n=k$, the statement still holds true. Thus, Lemma 3 is indeed the generalization of Lemma 2.

From the proof of Theorem 1.1, we notice that as long as the sum of unit vectors that each of them is perpendicular to the corresponding side is a zero vector, the polygon will possess CVS property.

It was hypothesized that the converse of the above statement also holds true. The hypothesis was articulated into Theorem 1.2. In the attempt to prove Theorem 1.2 holds true, it was noticed that the suggested proof contradicted to the previous work stating that:

"If V takes equal values at <u>TWO</u> distinct points, the polygon possesses CVS property''

(Elias Abboud, 2009).

Citing from the earlier proof of Theorem 1.2, from our results:

$$V(P) - V(Q) = \sum_{i=1}^{n} \overrightarrow{PA_\iota} \cdot \vec{e}_i - \sum_{i=1}^{n} \overrightarrow{QA_\iota} \cdot \vec{e}_i = \overrightarrow{PQ} \cdot \left(\sum_{i=1}^{n} \vec{e_\iota} \right) = \vec{0} \; (*)$$

Since P and Q are two distinct points, $\overrightarrow{PQ}$ is not a zero vector. However, from here, we cannot conclude that the sum of unit vectors that are perpendicular to the sides of the polygon is a zero vector even though vector *PQ* is a non zero vector. Noted that this conclusion is the key condition for a convex polygon to have CVS property.

We had concluded that because the given polygon possessed CVS property, the (*) hold true. Furthermore, because '$\overrightarrow{PQ}$ is not a zero vector' but the multiplication results in a zero vector, the other factor, which is $\vec{e}$, the sum of unit vectors, is infact a zero vector (**).

Yet this is a fast-concluding statement. From this deduction (**), it is able to prove that a polygon will possess CVS property as long as there exists **two** distinct points that take equal values of function *V*, which is much different from the results of Elias Bound (2009).

Thus, the proof is taken into consideration to further look into the problem and explain why there is such kind of difference.

With two distinct interior points, *P* and *Q*, since the polygon has CVS property:

$$V(P) = V(Q)$$

$$\Rightarrow \sum_{i=1}^{n} \overrightarrow{PA_\iota} \cdot \vec{e_\iota} - \sum_{i=1}^{n} \overrightarrow{QA_\iota} \cdot \vec{e_\iota} = \sum_{i=1}^{n} \overrightarrow{PQ} \cdot \vec{e_\iota} = \overrightarrow{PQ} \cdot \left(\sum_{i=1}^{n} \vec{e_\iota}\right) = \vec{0} \; (*)$$

From (*), the deduction should be: $\sum_{i=1}^{n} \vec{e_\iota} = \vec{0}$ (7.1)

Or $\sum_{i=1}^{n} \vec{e_\iota}) = \vec{e}$ and $\vec{e} \perp \overrightarrow{PQ}$ (7.2)

When $\overrightarrow{PQ}$ is perpendicular to the sum-vector. This results in $\overrightarrow{PQ} \cdot \sum_{i=1}^{n} \vec{e} = 0$, regardless of whether or not the polygon has CVS property.

That is, we cannot conclude that the sum of unit vectors is a zero vector even though vector *PQ* is a non zero vector since there is situations that is misleading and special cases where the equation obviously holds true without neither vector *PQ* nor sum of unit vectors necessarily being a zero vector. This is when $\overrightarrow{PQ}$ is perpendicular to $\vec{e}$, the sum-vector. This results in $\overrightarrow{PQ} \cdot \sum_{i=1}^{n} \vec{e} = 0$, regardless of whether or not the polygon has CVS property

If given a convex polygon, there are possibilities that the sum of unit vectors is a non-zero vector. Choosing two random interior points may accidentally form a vector whose direction perpendicular to the sum-vector.

To control the condition *(7.2)*, we first need to calculate the sum-vector before controlling PQ such that it is not parallel to this sum-vector. Since we have already calculated the sum-vector, our attempt to prove that the sum of unit vectors of this given polygon becomes meaningless.

From the earlier solution, the use of TWO points could not give us a direct and accurate explanation due to the existence of some cases that mislead our deduction. Therefore, we are going to increase the number of interior points from 2 points to 3 points for tighter conditions. As the number of conditions increased with a tighter condition, the conclusion that the polygon possesses CVS property can be made. Therefore, our approach explains the result of the extension of Viviani's theorem made by Elias Abboud (2009) that why there is a need for 3 non-collinear points that take the same value of *V(x)* for a polygon to posses CVS property.

The process to prove the hypothesis of Theorem 1.2, which is Theorem 1.3, does not only help to clarify the extension of Elias Bound (2009) but also to explain why some special polygons have CVS property (Chen, Zhibo; Liang, Tian. 2006). Those special polygons include: regular polygons, parallelograms, polygons with pairs of opposite sides parallel. It is discovered that those special polygons posses CVS property, most of the proof was based on using area-formula (Chen, Zhibo; Liang, Tian. 2006). From our hypothesis and attempt to prove it, we figured out some properties that a polygon would have, given the condition that it possesses CVS property. The investigation for the study was articulated into Theorem 1.4. In the study of Theorem 1.4, we deduced some of the possible shapes of the polygon. Those possible shapes match with the previous work stated above. To conclude, from Theorem 1.4, we are able to explain why certain geometrical properties of a polygon are needed so that the polygon would have constant *V(x)* sum.

Corollaries

In the process of approaching Viviani's theorem using vectors, some interesting corollaries related to Viviani's theorem have been discovered, all of which involves a constant sum of multiplication of vectors in a polygon.

The multiplication of vectors involve vector $\overrightarrow{A_\iota A_{\iota+1}}$ which is the vector of side of the polygon and $\overrightarrow{PB_{\iota+1}}$, which is the vector of the length from the given interior point P to sides of the polygon. The significance of Corollary 1 is that the sum of multiplication of vectors is a constant that involves the area of the polygon and the size and cosine of exterior angle of this given regular n-sided polygon ($\cos\left(90° - \frac{360°}{n}\right)$). Therefore, a new vector-related result is established, all of which matches the aim of the project that is to approach Viviani's theorem with another method which is to merely based on vectors.

The result of the corollary has a greater significance with the use of another random interior point O. This is because it is noted that P is indeed the converging points of all the perpendicular lines to the corresponding sides of the polygon. In other words, P is a special point inside the polygon. Yet when replacing P by another point O inside the polygon, the result still holds true. This is due to the special property of vectors, especially the arithmetic calculation like addition and that if segment was used instead, the result would no longer hold the same. Therefore, the result yielded for the corollary applies for not only the special converging point P but also for any interior points of the polygon given. Thus, a stronger result was obtained.

Yet will the converse of Corollary 1 still hold true? In other words, given the result of Corollary 1, is it possible to conclude that the polygon is regular? It was hypothesised that the converse of Corollary 1 still holds true. Thus, Corollary 2 is suggested as the converse of Corollary 1. The aim of Corollary 2 is to prove that the given polygon is regular, so we are able to conclude that it possesses CVS property with the below intended process:

1. Given the polygon with the assumption (*) of Corollary 2, prove that it is regular.
2. Since the polygon is regular, it possesses CVS property.

$$\sum_{i=1}^{n} \overrightarrow{A_iA_{i+1}} \cdot \overrightarrow{PB_{i+1}} = \sum_{i=1}^{n} \overrightarrow{A_iA_{i+1}} \cdot \overrightarrow{OB_{i+1}} = 2\, area\, of\, polygon \times \cos(90° - \frac{360°}{n}) \text{ (***)}$$

To prove a polygon regular, it is necessary to prove that it is both equiangular and equilateral. Specifically, there are two distinct variables in a polygon which is the size of interior angles and length of sides. It is noted that if those 2 variables are hidden at the same time, with only 1 equation (***), it is quite impossible to work the problem out. Therefore, condition that needs to be given for Corollary 2 was taken into consideration such that it would be possible to deduce a feasible solution when exploiting the assumption.

The assumption given for the polygon in Corollary 2 is that it is either an equilateral polygon or equiangular polygon. Yet when considering the final aim of the work which is to prove that it possesses CVS property, it is more preferred to make the polygon to be equilateral than equiangular. By constructing parallel lines, an equiangular polygon will become a regular polygon (Literature Review...) Therefore, our attempt to prove that the equiangular polygon will have constant V sum using the give assumption (***) will be less significant than constructing parallel lines to produce a regular polygon, which certainly possesses CVS property. Thus, the polygon in Corollary 2 is made to be equilateral instead.

As we attempted to produce a feasible solution for Corollary 2, there remains an unsolved challenge. We believed that the once this challenge is completely addressed, the Corollary 2 will be no longer a question for us to wonder and thus, our hypothesis could be clarified if it is confirmed to be true.

II. Carnot's theorem

Extended Carnot's theorem

It is noted that the model of Carnot's theorem in triangles is quite similar to that of Viviani's theorem. However, Carnot's theorem has not been extended to polygons like Viviani's theorem. It is noticing that the proof of Carnot's theorem is merely based on Pythagoras theorem; thus, it is hypothesizing that it would be possible to extend the theorem from triangles to polygons since a polygon could be divided into triangles from the given interior point.

The original Carnot's theorem holds true both forward and reverse ways. Similarly, when extending the original theorem, the extension also holds true for both ways. In fact, the similarity between the model of Carnot's theorem and that of Viviani's theorem becomes clearer as we compare the extension version of both theorems. Both models consist of a convex polygon with an interior point with the respective perpendicular lines from that interior point to sides of the given polygon.

Relationship between Carnot's theorem and Viviani's theorem

From the observation of the similarity in the two models of the two extended theorems, we decided to establish a link of those two extended theorems by combining the results found previously.

The first problem is inspired from Corollary 1 and the extended Carnot's theorem. It shows a link between the Corollary 1 and the extended Carnot's theorem in polygons to produce a new problem that can be solved merely using the results found earlier. The solution of the suggested problem is built up from the earlier work of extending Carnot's theorem and the application of Corollary 1 using the idea of Viviani's theorem that a regular polygon possessing CVS property.

In a similar manner, Problem 2 is suggested from the inspiration of Corollary 2. If Corollary 2 holds true, the hypothetical Problem 2 also holds true. In fact, Problem 2 is the converse statement of the Problem 1 since Corollary 2 is the converse of Corollary 1. Similarly, the steps for the solution of Problem 2 is developed the same way as that of Problem 1.

Due to time constraint, Corollary 2 has not been solved completely. It is believed that if there was more time devoted to further research to look into the unsolved challenge, the results of Corollary would have a more significant impact. Thus, with the completion in the process of proving Corollary 2, the project would have clearer result and better analysis for the approach and the connection between Carnot's theorem and Viviani's theorem.

CONCLUSION

In conclusion, the two main objectives of the project is to approach Viviani's theorem and its extension using vectors and to introduce a connection inspired from the results of the earlier approach with another theorem in the same field, Geometry in Mathematics.

Overall, through a new way of approach towards the original Viviani's theorem and its extension, it is discovered that polygons that possess CVS property would have the sum of the unit vectors, such that each of them is perpendicular to its corresponding side, is a zero vector. This key finding of the project explained the rationale behind the result yielded of Elias Abboud, further investigated some special geometrical properties of Viviani polygons, thus, explained why some certain special polygons, such as regular polygons, parallelograms, polygons with opposite sides parallel would be CVS polygons.

Carnot's theorem, whose model is found to have a similarity with Viviani's theorem, is further studied and extended so that the link between those two theorems would be established, making the relationship between two theorems clearer. As the results, two problems (Link 1 and Link 2) have been introduced with the suggested solution merely developed from the all the earlier work of the project, especially the two corollaries specifically. However, there remains a challenge which is unsolved due to time constraint, which is believed to make the project be more inclusive and successful if it is completely worked out.

In future work, the project could be further investigated in three-dimension since vectors are a strong tool in Geometry. The Corollaries found could hold true even when generalized from 2D to 3D. The results, if held true in both 2D and 3D, would leave a larger impact and be more significant compared to the results that only hold true in two-dimensional space.

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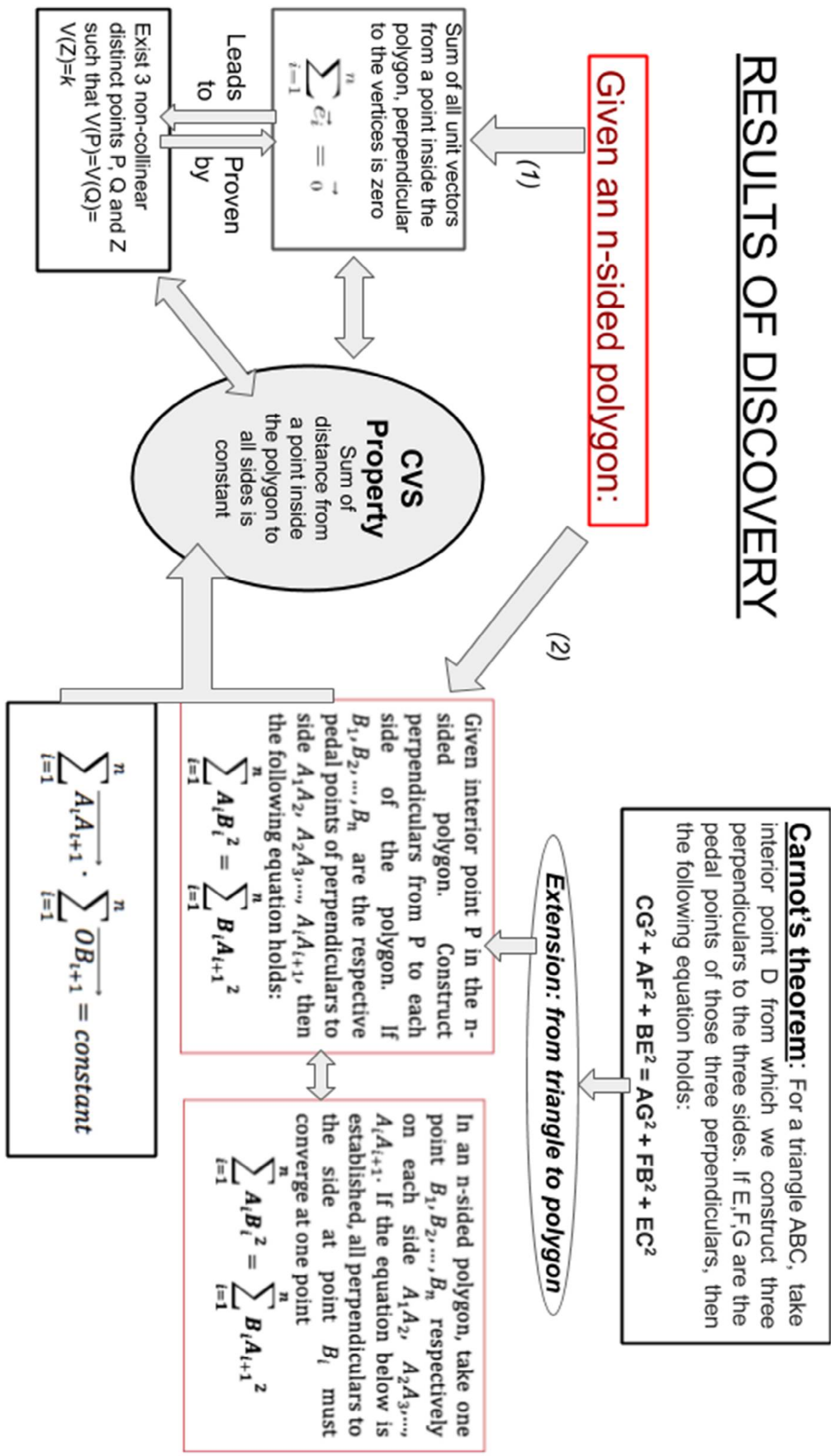
ANNEX


RESULTS OF DISCOVERY
Given an n-sided polygon:
(1)
Sum of all unit vectors from a point inside the polygon, perpendicular to the vertices is zero
$\sum_{i=1}^{n} \vec{e_i} = \vec{0}$
Leads to
Proven by
Exist 3 non-collinear distinct points P, Q and Z such that V(P)=V(Q)=V(Z)=k
CVS Property
Sum of distance from a point inside the polygon to all sides is constant
(2)
Given interior point P in the n-sided polygon. Construct perpendiculars from P to each side of the polygon. If $B_1, B_2, \dots, B_n$ are the respective pedal points of perpendiculars to side A_1A_2, $A_2A_3, \dots, A_iA_{i+1}$, then the following equation holds:
$\sum_{i=1}^{n} A_iB_i^2 = \sum_{i=1}^{n} B_iA_{i+1}^2$
$\sum_{i=1}^{n} \overrightarrow{A_iA_{i+1}} \cdot \sum_{i=1}^{n} \overrightarrow{OB_{i+1}} = constant$
In an n-sided polygon, take one point $B_1, B_2, \dots, B_n$ respectively on each side A_1A_2, $A_2A_3, \dots, A_iA_{i+1}$. If the equation below is established, all perpendiculars to the side at point B_i must converge at one point
$\sum_{i=1}^{n} A_iB_i^2 = \sum_{i=1}^{n} B_iA_{i+1}^2$
Carnot's theorem: For a triangle ABC, take interior point D from which we construct three perpendiculars to the three sides. If E,F,G are the pedal points of those three perpendiculars, then the following equation holds:
$CG^2 + AF^2 + BE^2 = AG^2 + FB^2 + EC^2$
Extension: from triangle to polygon