The Boomerang Theorem

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Abstract

The Boomerang Theorem is a geometrical theorem derived and expanded from the concept of the Simson line and the Thales Theorem. The Boomerang Theorem involves four Simson Lines that interact to produce interesting relationships we explored. These four lines form a boomerang-like figure leading us to name the results of our study the Boomerang Theorem. In this report, we shall explore this newly derived theorem and understand the relationships involved. However, as we had a time constraint to finish our research, there are still numerous expansions ready to be explored on this Boomerang Theorem.

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I. Introduction

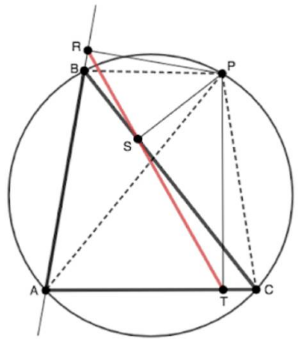
At the beginning, we found the Simson line theory while we were researching for our project. We did some research on it to understand the theory. From the Simson Line theory, we reapplied the concepts and conditions of Simson Line on a quadrilateral instead of a triangle. As we make changes to the diagram of the combination of the Simson Line theory and quadrilateral, we realised that the magnitude of the angles on the diagram had properties and relationships to be studied. We moved on to find the proofs and try to understand the whole theory. We extended the Simson Line and found the Boomerang Theorem, and we used the Thales' Theorem to prove it.

Our project focuses mainly on the relationships of the four Simson Lines created, which formed a boomerang-shaped figure and the angles formed in the whole diagram, and also to find out the reasons why the magnitude of the angles remains the same even when we move the earth point around.

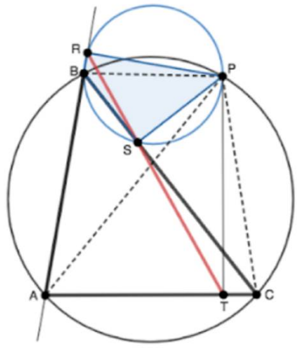
II. Literature Review

a) Simson Line

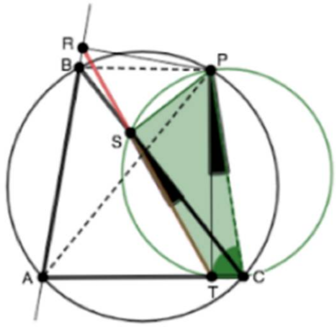
We did some research and understood the theory behind the Simson Line. In geometry, given a triangle *ABC* and a point *P* on its circumcircle, the three closest points to P on the lines *AB*, *AC*, and *BC* are collinear. The line through these points is the Simson line of *P*, named for Robert Simson. We studied the proof provided. Given a triangle *ABC* and point *P* which lies on the circumcircle of the triangle. First, we construct a dotted line from *P* to the three vertices of the triangle, *A*, *B* and *C*.



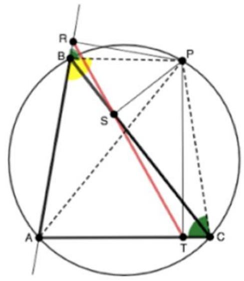
Angle *BRP* equals to angle *BSP* which is 90°. *RBSP* is a cyclic quadrilateral. The angle sum of triangle equals to 180°, which is the total of angle *BRP*, angle *BPR*, and angle *RBP*. The sum of angle *RBP* and angle *BPR* equals to 90° too.



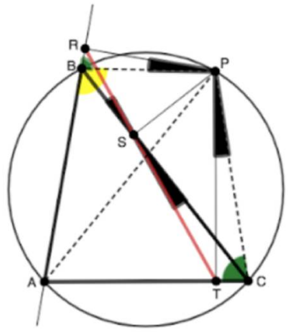
Similarly, angle *PST* equals to angle *PTC*, which is 90°. *PSTC* is a cyclic quadrilateral. The sum of angle *TPC* and angle *TCP* is 90°.



The sum of angle *ABP* and angle *ACP* is 180° as opposite angles in a cyclic quadrilateral are supplementary. Angle *RBP* equals to angle *TCP*. Angle *TPC* equals angle *BPR* as angle *RBP* equals to angle *TCP.*



Angle *BRP* equals to angle *BSP*, which is 90°. Hence, angle *BSP* equals to angle *TSC* as angles in the same segment are equal. *RST* is a straight line and vertically opposite angles are equal. BC in one of the sides of the triangle *ABC.*



From the original theory of Simson Line, we understood that the three closest point the point at the circumference are collinear. We further expanded this theory by changing the triangle, into an irregular quadrilateral. From the diagram, we realized that there will be four triangles formed in the quadrilateral and thus we can form four Simson Lines, one on each triangle.

b) Thales's Theorem

When we were trying to prove our Boomerang Theorem, we used Thale's theorem. In geometry, Thales's theorem states that if *A, B*, and *C* are distinct points on a circle where the line *AC* is a diameter, then the angle ∠*ABC* is a right angle. One of the proves used the fact that the sum of the angles in a triangle is equal to 180° and the base angles of an isosceles triangle are equal. Given a circle which O is the centre while *A*, *B* and *C* are points on the circumference and straight-line *AC* is the diameter. Since *OA*, *OB* and *OC* are equal in length, *OBC* and *OBA* are isosceles triangles and angle *OBC* equals to angle *OCB*, angle *OBA* equals to angle *OAB*. Solving the algebraic equation, we know that the sum of one of the base angles of triangle *OAB* and one of the base angles of triangle *OBC* is 90°.

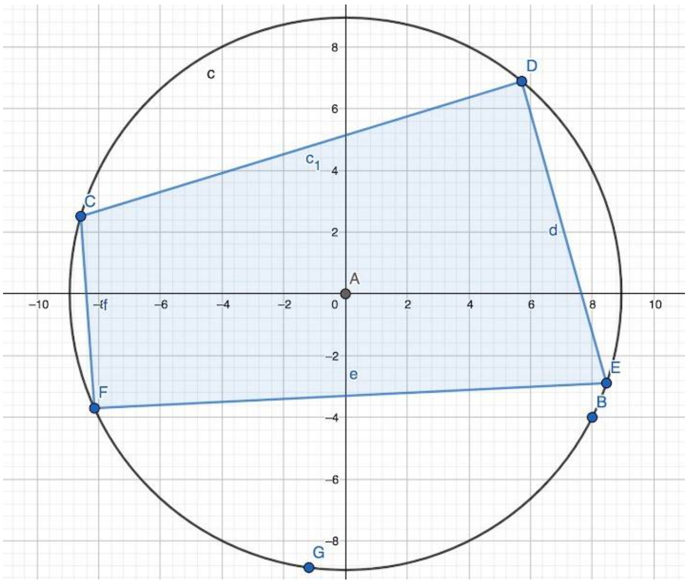
III. Methodology

On a cyclic quadrilateral of any size, drop a random point- the Earth Point- on the circumference of the circle formed. A perpendicular line can be created from the point to all four sides and the diagonals of the quadrilateral. The intersections of the perpendicular lines and each of the four sides and its diagonals will then be able to form a boomerang-shaped quadrilateral once connected.

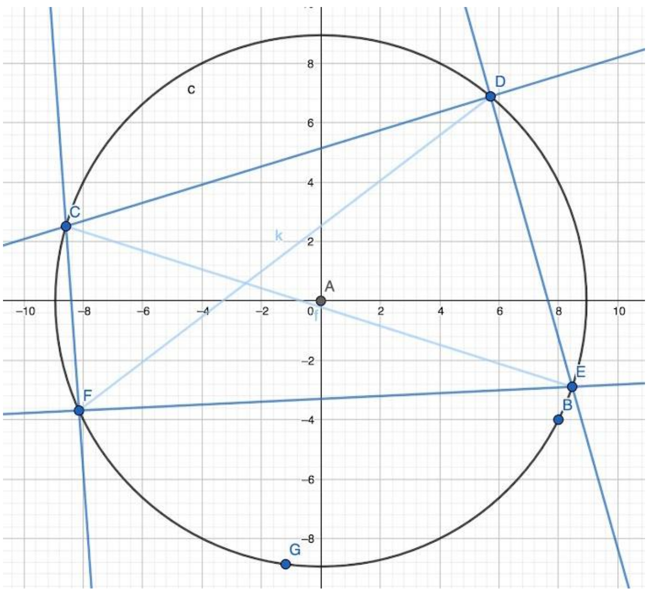
Steps to construct:

1. Construct any random circle.

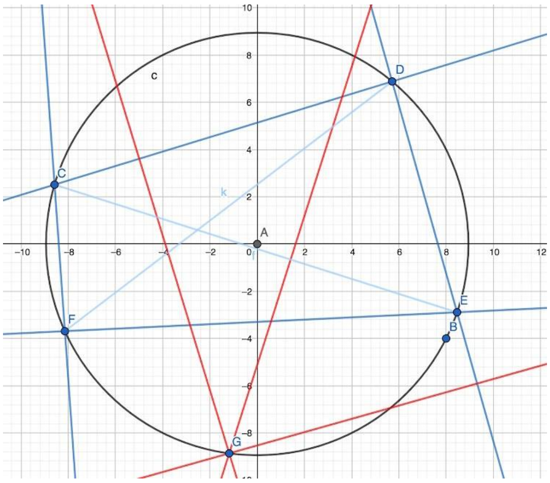
2. Create any quadrilateral with its four vertices on the circumference of the circle. The quadrilateral can be either regular or irregular. Pick and mark a point anywhere on the circumference. This point is known as the Earth Point (Point G).



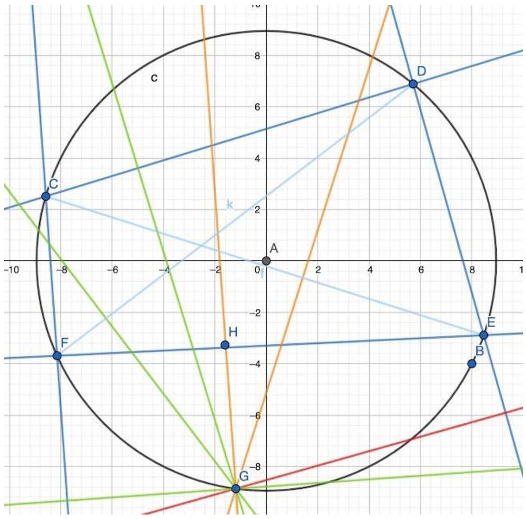
3. Extend the sides of the quadrilateral with straight lines. There will be four triangles in the quadrilateral: ΔCDE, ΔCFE, ΔCDF and ΔDEF such that the vertices of each triangle is three of the four vertices of the quadrilateral.



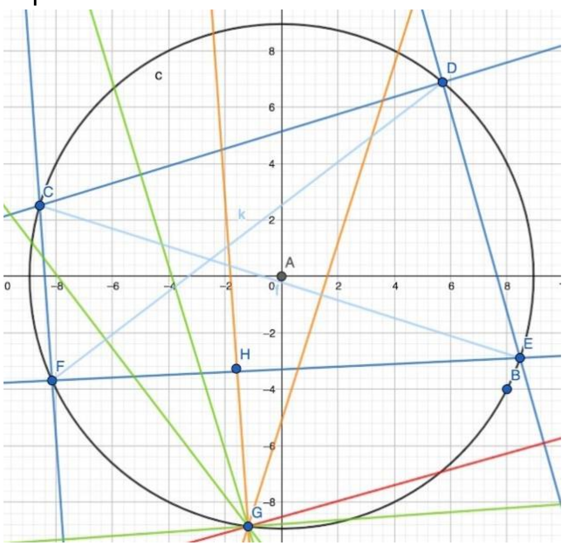
4. Drop a perpendicular line from point G to each sides of $\triangle$CDE.



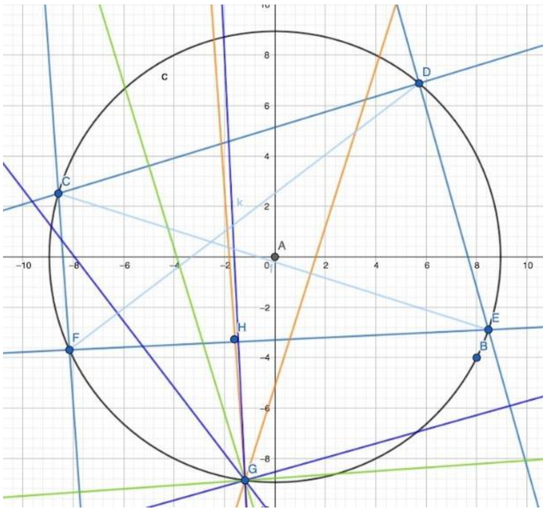
5. Drop a perpendicular line from point G to each sides of $\triangle$CFE. One of the perpendicular lines from the sides of $\triangle$CDE to point G overlapped with one of the perpendicular lines from the sides of $\triangle$CFE to point G.



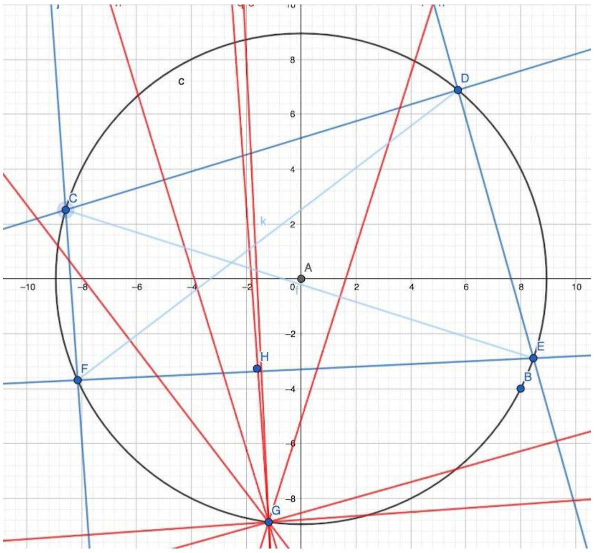
6. Similarly, Drop a perpendicular line from point G to each sides of $\triangle$CDF. Two of the lines overlapped with the previous perpendicular lines created.



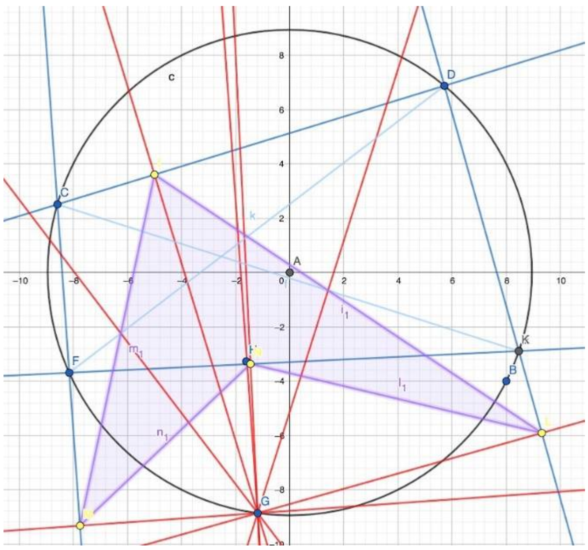
7. Drop a perpendicular line from point G to each sides of ΔDEF. Two of the lines overlapped with the previous perpendicular lines created.



8. The final view of the diagram after constructing all the perpendicular lines will be as below.



9. The intersection points of the perpendicular lines from the sides of quadrilateral to point G are marked out and joined together. A boomerang-shaped polygon is formed.

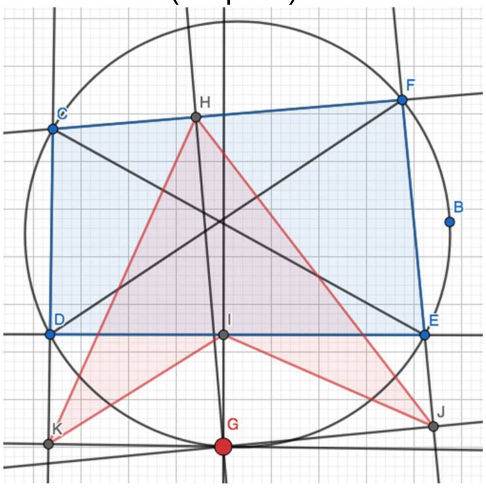


IV. Findings

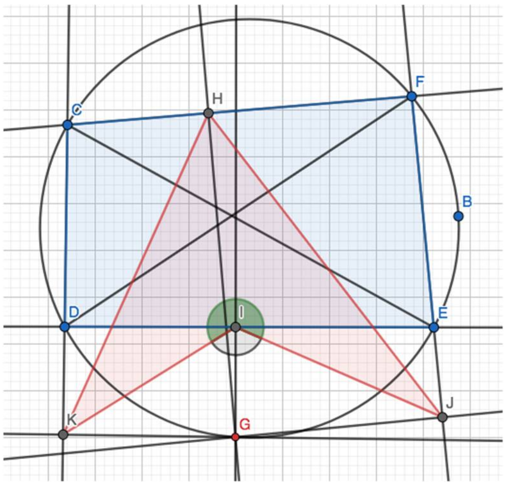
a) Properties of The Boomerang

For the ease of explaining this theorem, we decided to give the five important angles in boomerang theorem names.

1. Earth Point (red point)

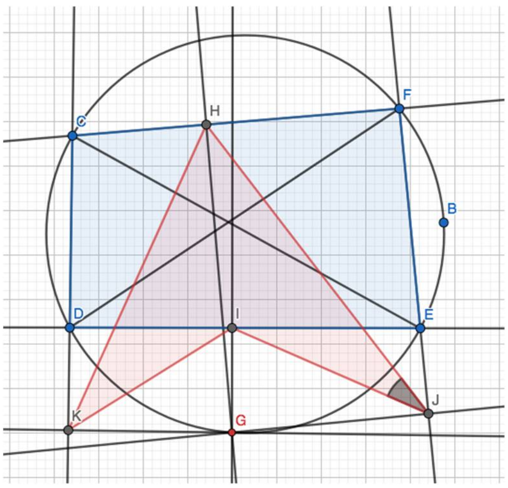


2. Yakt Angle (∠KIJ)



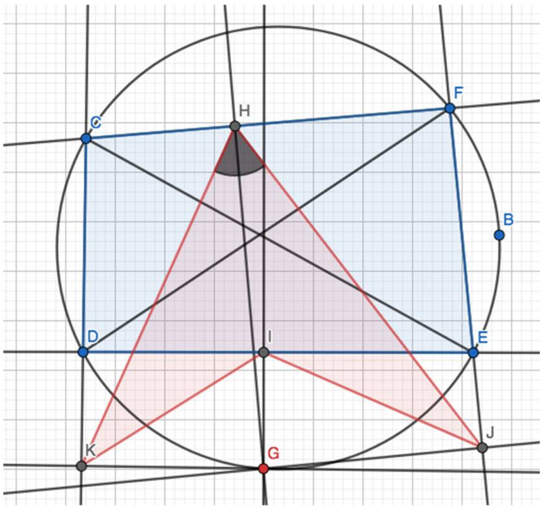
The Yakt angle is divided into two parts :
Minor Yakt angle (grey) and Major Yakt angle (green)

3. Leading Angle (∠IJH)



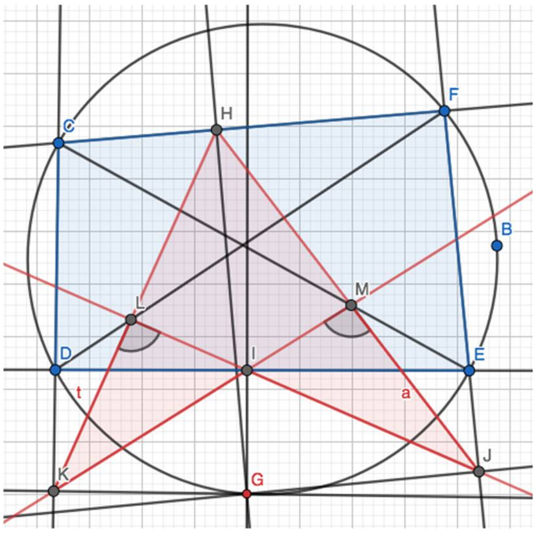
The name 'Leading' is inspired by the name of the right wing of a boomerang.

4. Xylie Angle (∠KHJ)



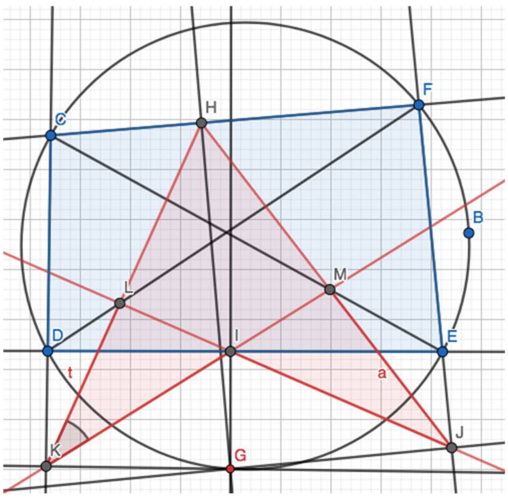
The name 'Xylie' is derived from Western Australian which translated boomerang to 'Kylie'

5. Supporter Angles



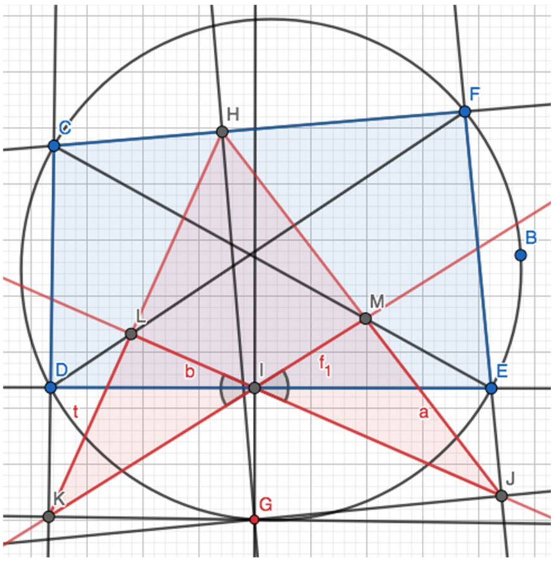
Left supporter angle = ∠JLK
Right supporter angle = ∠KMJ

6. Trailing Angle (∠HKI)



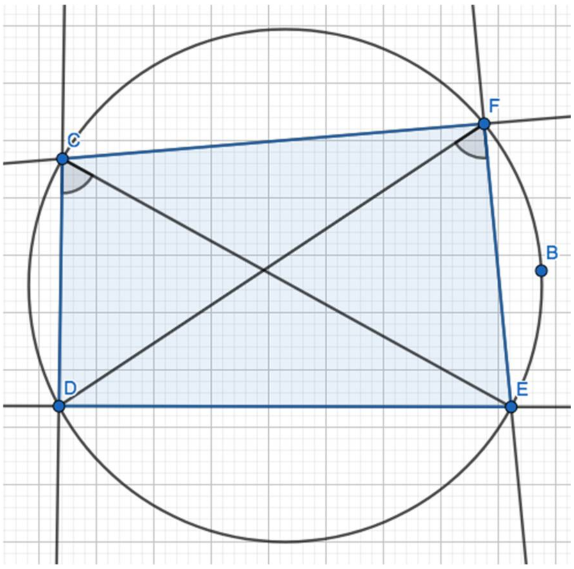
The name 'Trailing' is inspired by the name of left wing of a boomerang.

7. Elk Angle



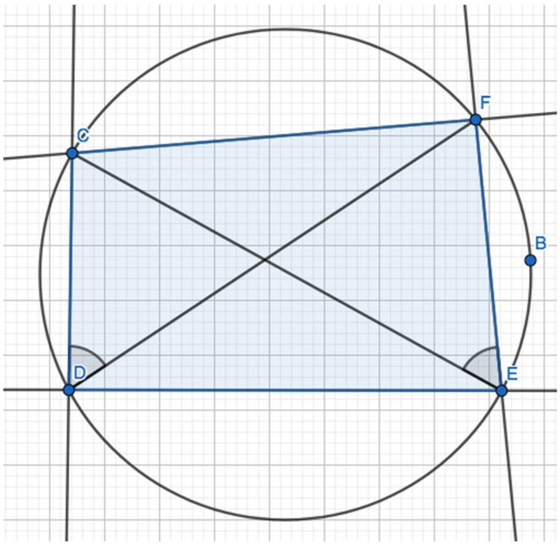
Left Elk Angle = ∠LIK
Right Elk Angle = ∠MIJ

8. Himalaya Angles



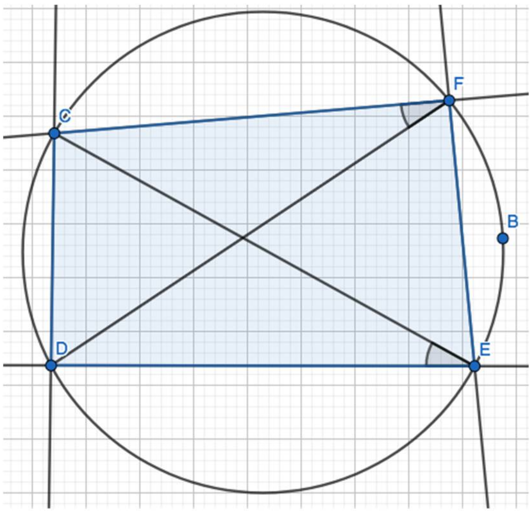
Left Himalaya Angle = ∠DCE
Right Himalaya Angle = ∠DFE

9. Mariana Angles



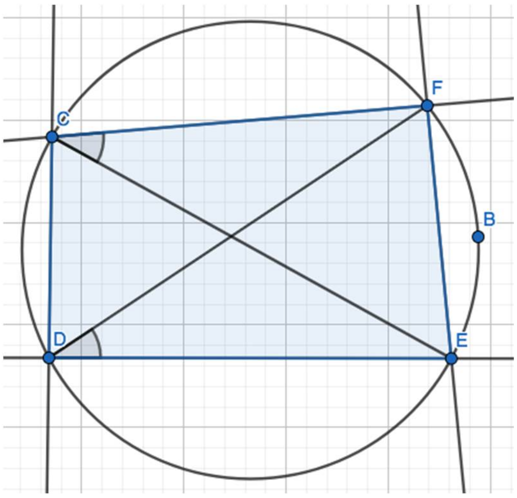
Left Mariana Angle = ∠CDF
Right Mariana Angle = ∠CEF

10. Prav Angles



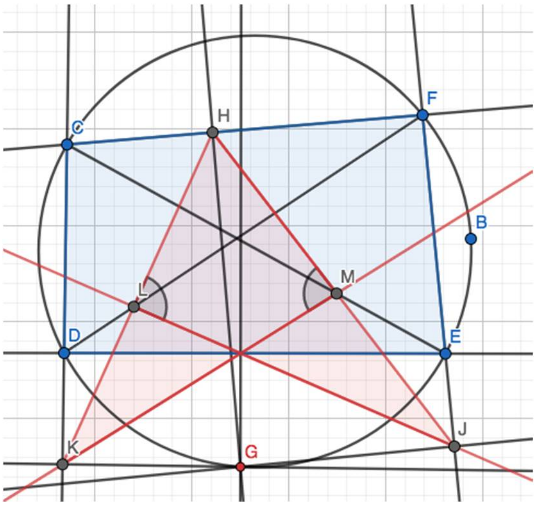
Upper Prav Angle = ∠CFD
Lower Prav Angle = ∠CED
The word 'Prav' is the Slovenian word for 'right'.

11. Levo Angles



Upper Levo Angle = ∠FCE
Lower Levo Angle = ∠FDE
The word 'Levo' is the Slovenian word for 'left'.

12. Finn Angles



Left Finn angle = ∠HLJ
Right Finn angle = ∠HMK

The Properties of the Boomerang

The following pictures are for better referencing and visualisation of the relationship of the angles to avoid any misconception or confusion. Image 1 and 2 are of the same quadrilateral but different locations of the earth point with the consequent changes in the positions of the Simson lines.

Image 1

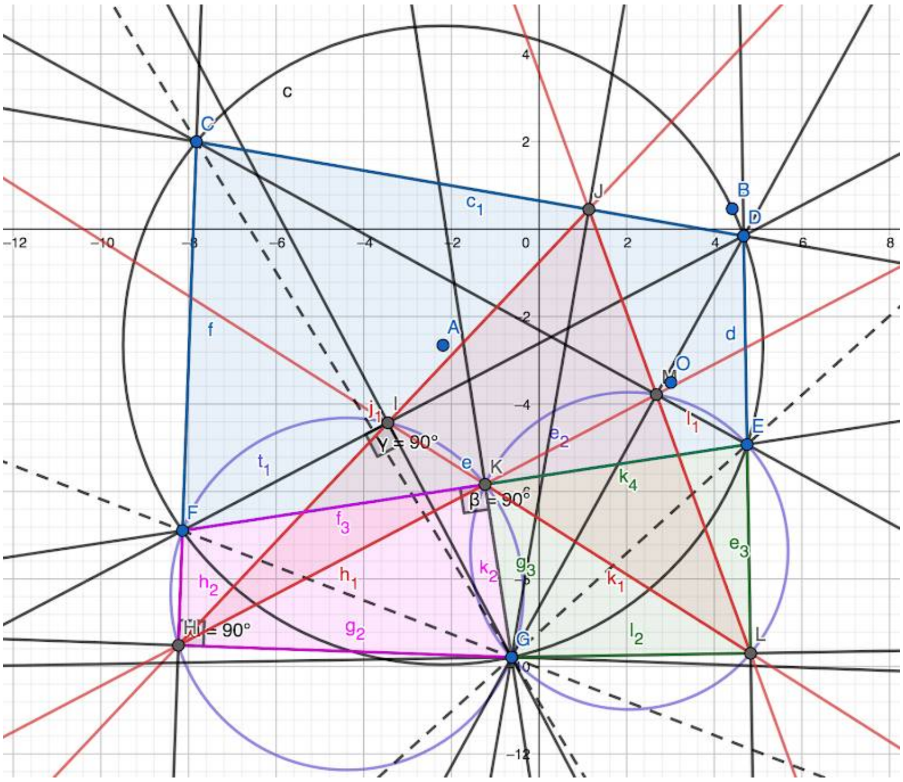
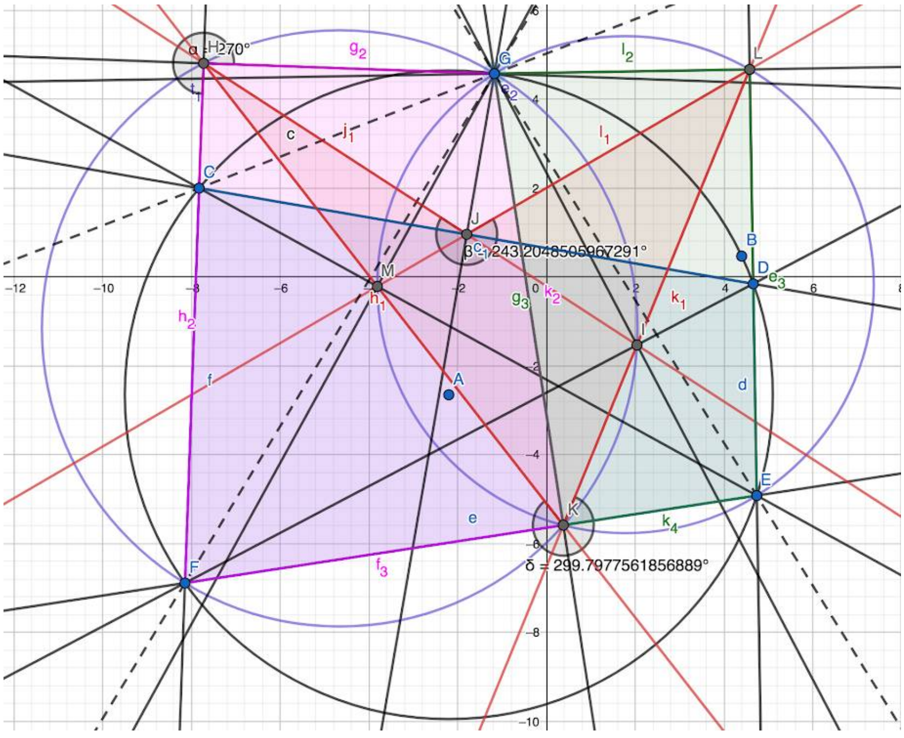


Image 2



1. Xylie angle will always be equal to the Himalaya angles in the quadrilateral which corresponds to it. The two angles are exactly facing the same side as the xylie angle. E.g. ∠HJL=∠FCE=∠FDE

2. Trailing and Leading angle will always be equal to the 2 corresponding angles to it. E.g. ∠JHK=∠DFE=∠DCE
   ∠JLK=∠CEF=∠CDF

3. Two parts of the Yakt angle which are equal to each other are also equal to the angles in the quadrilateral which point to the opposite direction to which the boomerang is pointing to.
   E.g. ∠IKH=∠MKL=∠CFD=∠CED

4. The minor Yakt angle is supplementary to itself after moving the earth point to the other side of the circle.
   E.g. Minor ∠HKL IMAGE1+ Minor ∠HKL IMAGE 2=180

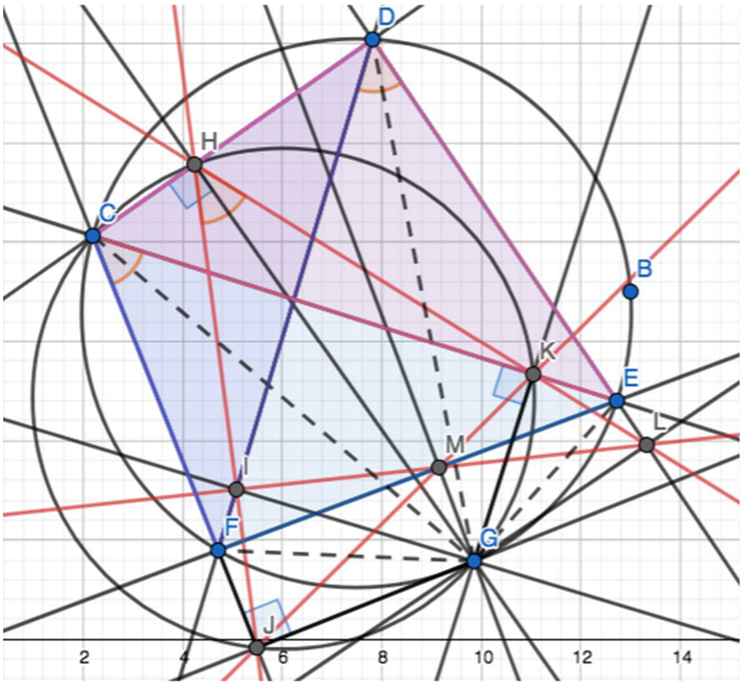
5. The four supporter angles are equal to the four angles in the quadrilateral.
   E.g. ∠HIK =∠CDE
   ∠LMK=∠FCD
   ∠JIK =∠CFE
   ∠JMK=∠DEF

6. The triangles in the boomerang can form cyclic quadrilaterals which pass through the Earth Point.
   E.g. CJOH, HFIK, HFKG, CJMH, DJIL, LEMK and LEKG

7. Every angle in the boomerang (xylie, trailing, leading, supporter, yakt, etc) will remain the same in magnitude despite the changes made to the position of the earth point. The magnitude of these angles can only be altered when there are changes in the shape of the original quadrilateral which cause alteration in some of the angles (Himalaya, Marina, Prav and Levo angles)

B) Proof of the Properties of the Boomerang

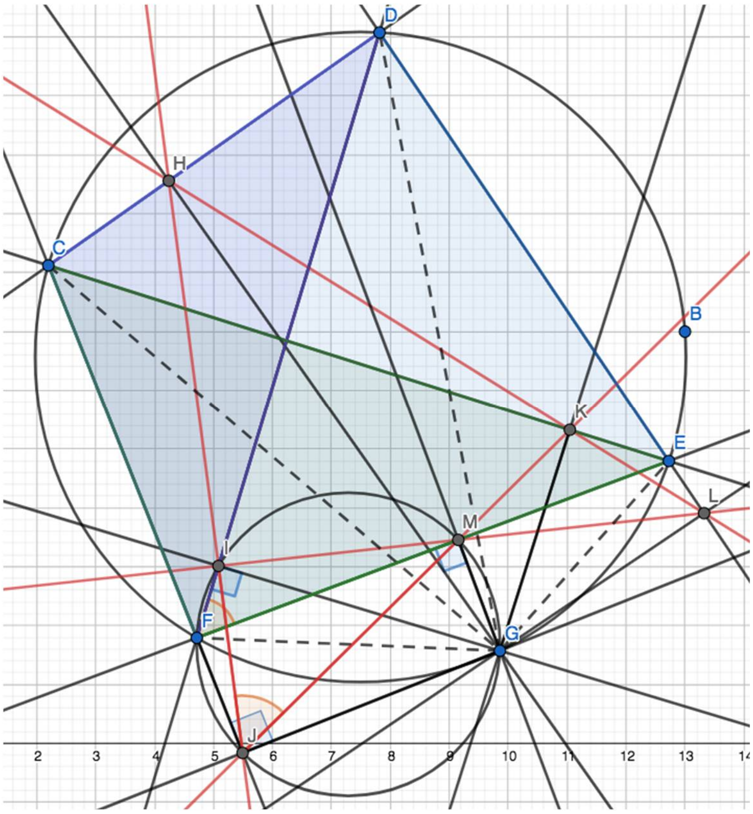
1. The Xylie angle is equal to the Himalaya angles
(∠ JHL= ∠ FCE= ∠ FDE)



∠GHC= 90°
∠GJC = 90°
∠GKC= 90°
∴By the converse of Thales' Theorem, vertices G, K, H, J and C lie on the same circle

∠JHK=∠JCK (angles in the same segment are equal)
∠LHJ=∠ECF
∴∠JHL=∠FCE=∠FDE

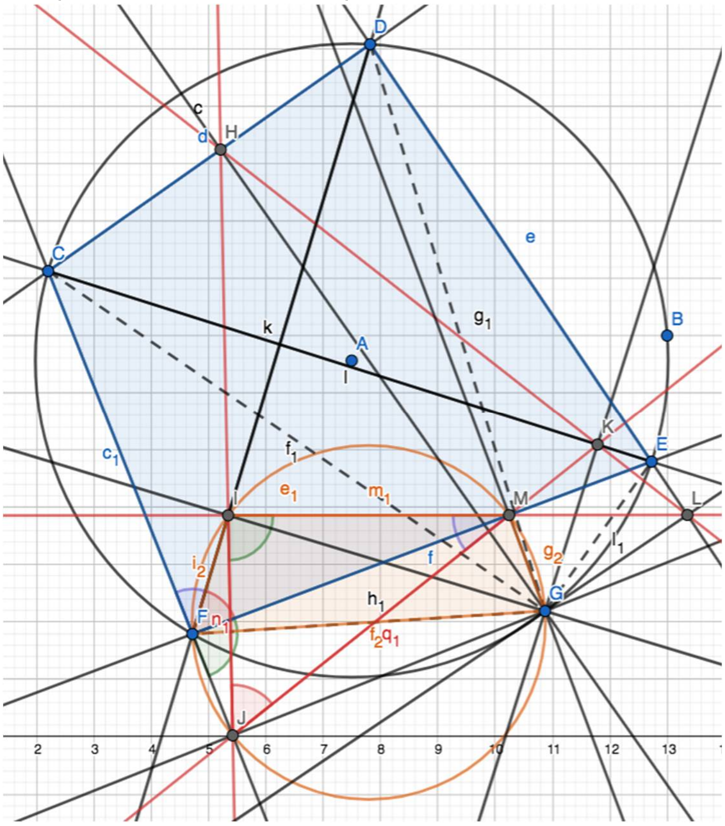
2. The Trailing angle is equal to the Levo angles
(∠HJM = ∠DFE = ∠DCE)



JIH and JMK are the simson lines from ΔDFC and ΔECF.
∠GIF= 90°
∠GJF=90°
∠FMG=90°
∴By Thales' Theorem, vertices I, F, J, G, M lie on the same circle

∠HJK = ∠DFE (angles in the same segment, ∠IJM = ∠IFM)
∠DFE = ∠DCE (angles in the same segment)
∴∠HJM = ∠DFE = ∠DCE

3. The Elk angles are equal to the Mariana angles (HJM=HLM=DFC=DEC)



The boomerang HJML is formed by Circle A, quadrilateral CDEF, and point G on the circle
From the second proof , we know that vertices F, I, M, G, J lie on the same circle.

∠MIJ= ∠MFJ (angles in the same segment)

∠Trailing(MJI) =∠MFI

∠IMJ = 180 - ∠MIJ -∠Trailing(MJI) (angle sum of triangle)

And ∠LMJ = 180 - ∠IMJ = ∠MIJ + ∠Trailing(MJI)

∠MIJ + ∠IJM = ∠MFJ + ∠IFM

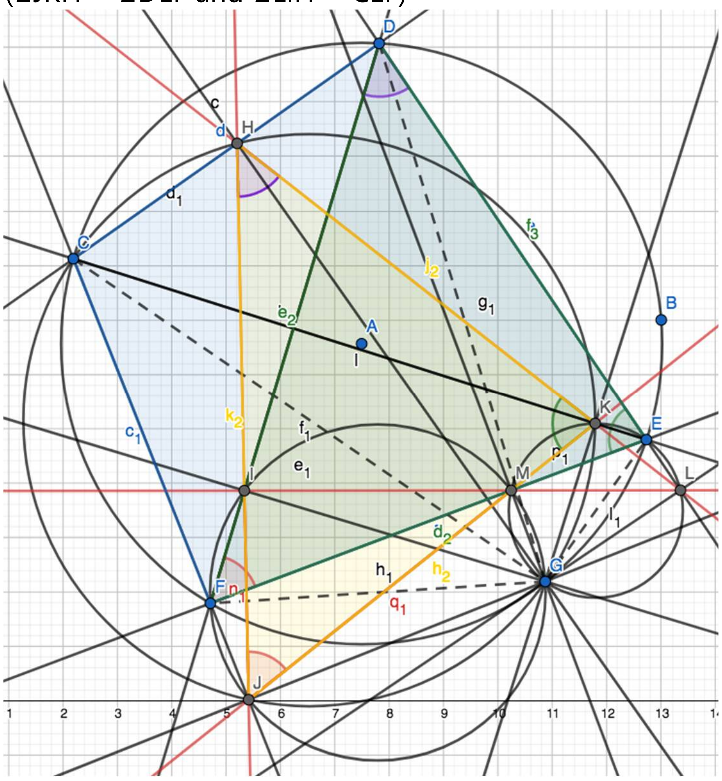
∠IMJ = 180 - ∠MIJ - ∠IJM

∠DFC = 180 - ∠MFJ - ∠IFM

∠IMJ = ∠DFC

With the same method we also can proof that ∠KML = ∠CED

4. The right Finn angle is equal to the right Mariana angle + lower Prav angle. The left Finn angle is equal to the left Mariana angle + lower Levo angle. (∠JKH = ∠DEF and ∠LIH = CEF)



The boomerang HJML is formed by Circle A, quadrilateral CDEF, and point G on the circle
From the other proofs we know that

∠MFI = ∠Trailing (MJI), ∠EDF=KHJ

∠MFI = ∠EFD

∠Trailing(MJI) = ∠KJH

∠DEF = 180 - ∠EDF - ∠EFD (angle sum of triangle)

∠JKH = 180 - ∠KHJ - ∠HJK (angle sum of triangle)

∠JKH = ∠DEF

With the same method we also can proof that ∠LIH = CEF

5. The minor Yakt angle (∠LMJ) in image 1 is supplementary to minor Yakt angle (∠LMJ) in image 2 when the point is brought to the opposite side of the quadrilateral.

Image 1

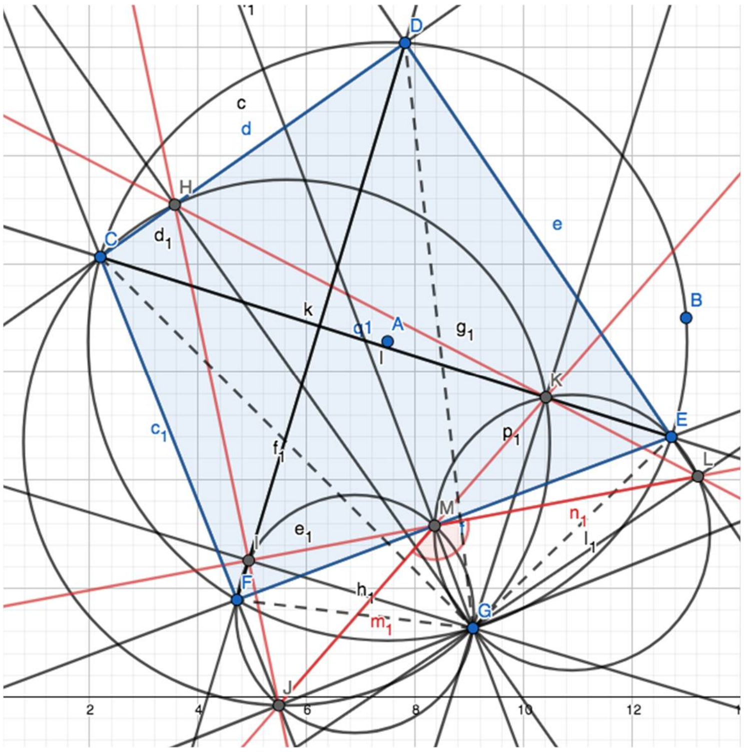


IMAGE 1
From Proof 3,
∠IMJ=∠DFC
∠LMJ (1) = 180-∠IMJ= 180-∠DFC

Image 2

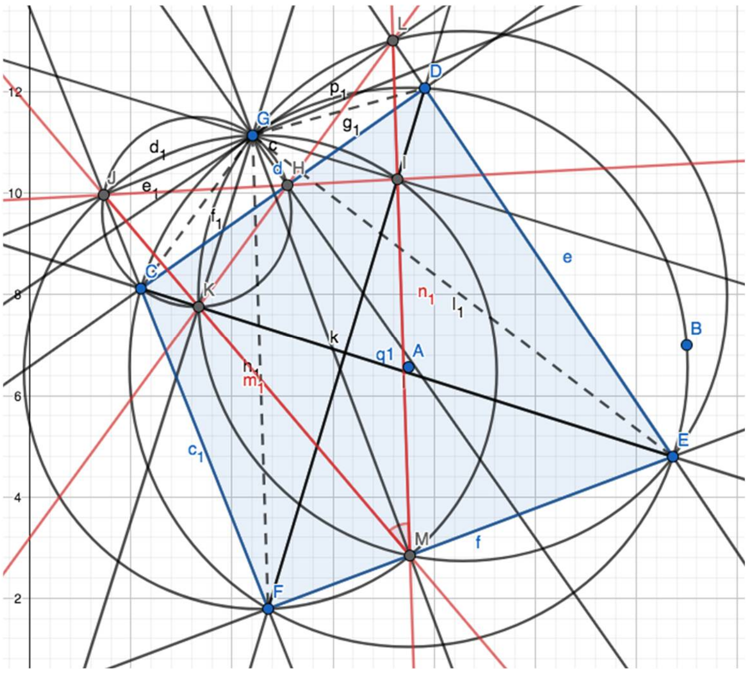
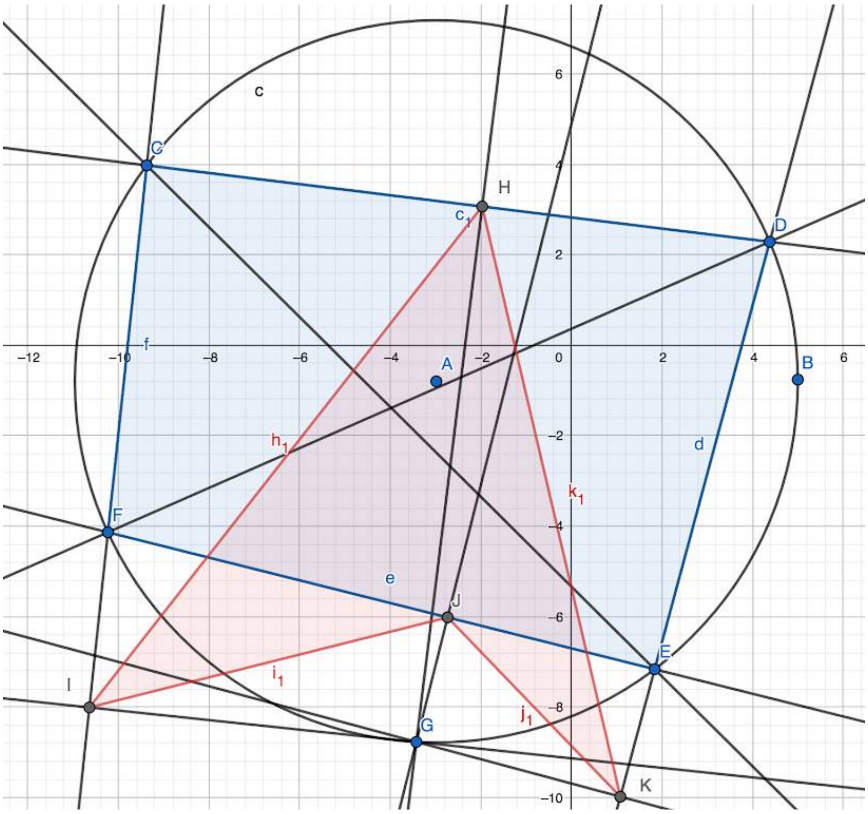


IMAGE 2
Applying Proof 1 on ∠LMJ,
∠LMJ (2) = ∠DFC = ∠DEC

C) Angles in the Boomerang

GIVEN THE LENGTHS OF THE SIDES OF A QUADRILATERAL, WHAT ARE THE ANGLES IN THE BOOMERANG?



The First Property of The Boomerang: *The Xylie angle is equal to the Himalaya angles* **(∠IHK=∠FCE=∠FDE)**

STEP 1: Find the lengths of the diagonals

Let CF=a, FE=b, DE=c and DC=d
Let the diagonals DF=q and CE=p

Length of DF

$$q = \sqrt{\frac{(ac+bd)(ab+cd)}{ad+bc}}$$

Length of CE

$$p = \sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}}$$

By Ptolemy's Theorem,

$$pq = ac + bd.$$

By Ptolemy's Second Theorem,

$$\frac{p}{q} = \frac{ad+bc}{ab+cd}$$

STEP 2 : Find the Magnitude of the angle

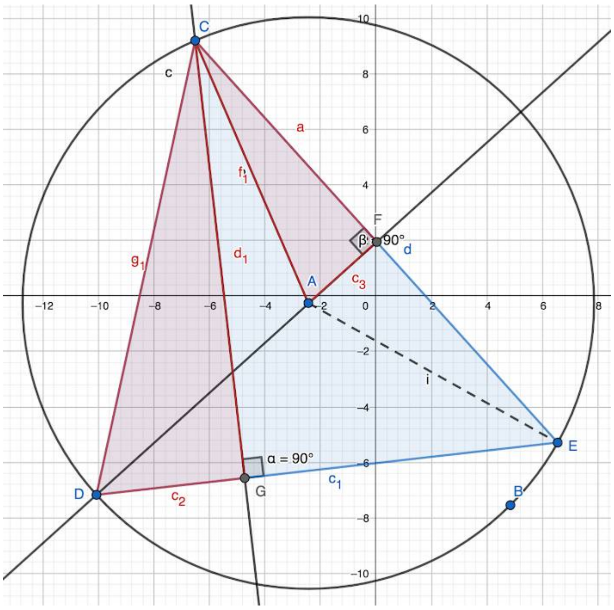
By Cosine Rule,

$$b^2 = a^2 + p^2 - 2(a)(p)cos\angle FCE$$

$$cos\angle FCE = \frac{a^2 + p^2 - b^2}{2ap}$$

PROOF OF: $q = \sqrt{\frac{(ac+bd)(ab+cd)}{ad+bc}}$

Shown below is visualisation of a triangle in its circumcircle to proof lemma 1 which says, "The area of a triangle is equal to the product of its sides divided by four times of its circumcircle's circumference".



LEMMA 1: Area of a triangle

Proof

In ΔCGD and ΔCFA,

∠CGD = ∠CFA = 90°

∠CDG = ∠CAF (central angle is twice any inscribed angle subtended by the same minor arc CE) (a line perpendicular to CE and passes through the centre of the circle bisects chord CE)

∴ ΔCGD is similar to ΔCFA (2 pairs of corresponding angles are equal)

Let R be the radius of the circumcircle of ΔCDE

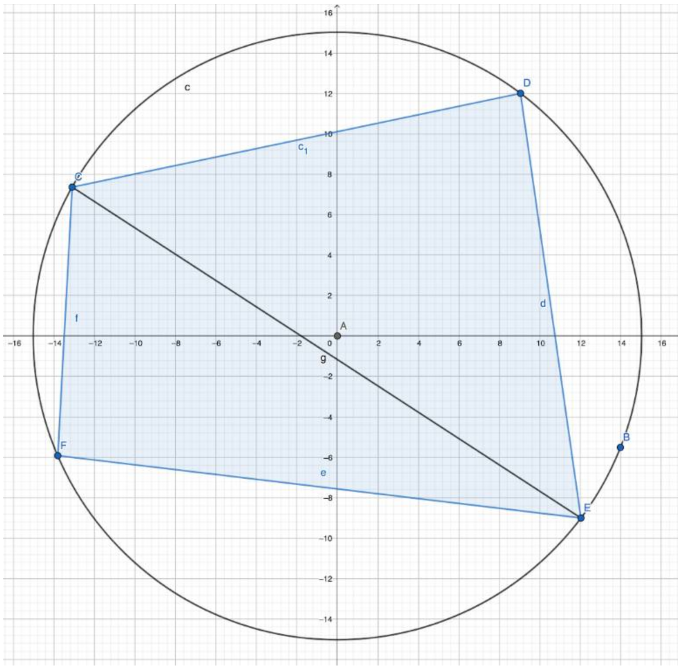
$$\frac{CG}{CD} = \frac{\frac{CE}{2}}{R}$$

$$CG = \frac{CE \times CD}{2R}$$

$$\text{Area of } \Delta CDE = \frac{1}{2} \times \frac{CE \times CD}{2R} \times DE$$

$$= \frac{CE \times CD \times DE}{4R}$$

1. Proof of Second Ptolemy's Theorem $\dfrac{p}{q} = \dfrac{ad+bc}{ab+cd}$



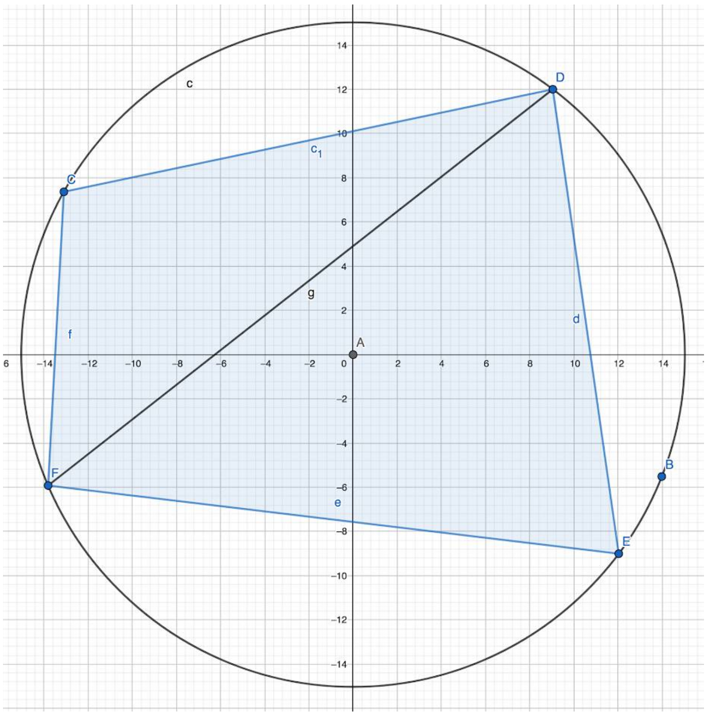
Let the area of the quadrilateral be Q
Let CF=a, EF=b, DE=c, CD=d and CE=p

By Lemma 1,

$$Q = \frac{abp}{4R} + \frac{pcd}{4R}$$

$$= \frac{p(ab+cd)}{4R}$$

→(Let this be the first equation)



Let the area of the quadrilateral be Q
Let CF=a, EF=b, DE=c, CD=d and DF=q

Using another diagonal and lemma 1,

$$Q = \frac{aqd}{4R} + \frac{bcq}{4R}$$

$$= \frac{q(ad+bc)}{4R}$$

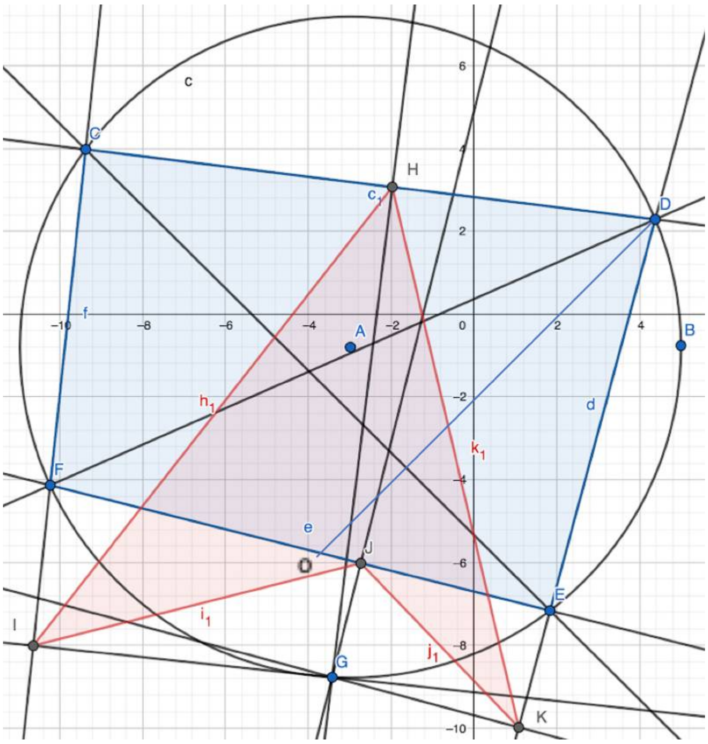
→(Let this be the second equation)

From equations 1 and 2,

$$p(ab + cd) = q(ad + bc)$$

$$\frac{p}{q} = \frac{ad+bc}{ab+cd}$$

2. Proof of $pq = ac + bd.$



Let CF=a, EF=b, DE=c, DC=d, CE=p and DF=q

The theorem will be proven when

$$(CE \times DF) = (CF \times DE) + (DC \times FE)$$

Construct line DO such that ∠ODE =∠CDF=∠CEF

From ▵CDF and ▵ODE,
∠CDF=∠ODE
∠CED=∠CFD (angles in the same segment are equal)
∴ ▵CDF and ▵ODE are similar (2 pairs of corresponding angles are equal)

Similarly, from ▵CDO and ▵FDE,

∠CDO=∠FDE
∠DCE=∠DFE (angles in the same segment are equal)
∴ ▵CDO and ▵FDE are similar (2 pairs of corresponding angles are equal)

$$\frac{OE}{GF} = \frac{DE}{DF}$$

$$OE \times DF = CF \times DE$$

$$\frac{CO}{FE} = \frac{DC}{DF}$$

$$CO \times DF = DC \times FE$$

Adding the two equations

$$(OE \times DF) + (CO \times DF) = (CF \times DE) + (DC \times FE)$$

$$DF(OE + CO) = (CF \times DE) + (DC \times FE)$$

$$(CE \times DF) = (CF \times DE) + (DC \times FE)$$

$$pq = ac + bd.$$

As the angles in the boomerang are actually equal to the angles of the triangles in the quadrilateral, the magnitude of xylie, trailing and leading angles can be simply found by using cosine rule. For yakt angle, properties of a 4-sided polygon can be used. The total angle in a 4-sided polygon is 360°. Hence, by simple subtraction, the magnitude of the yakt angle can be found.

From 1 and 2,

$$\frac{p}{q} = \frac{ad+bc}{ab+cd}$$

$$p = \frac{q(ad+bc)}{ab+cd}$$

Sub into 3

$$q \times \frac{q(ad+bc)}{ab+cd} = ac + bd$$

$$q^2 = \frac{(ac+bd)(ab+cd)}{ad+bc}$$

$$q = \sqrt{\frac{(ac+bd)(ab+cd)}{ad+bc}}$$

V. CONCLUSION

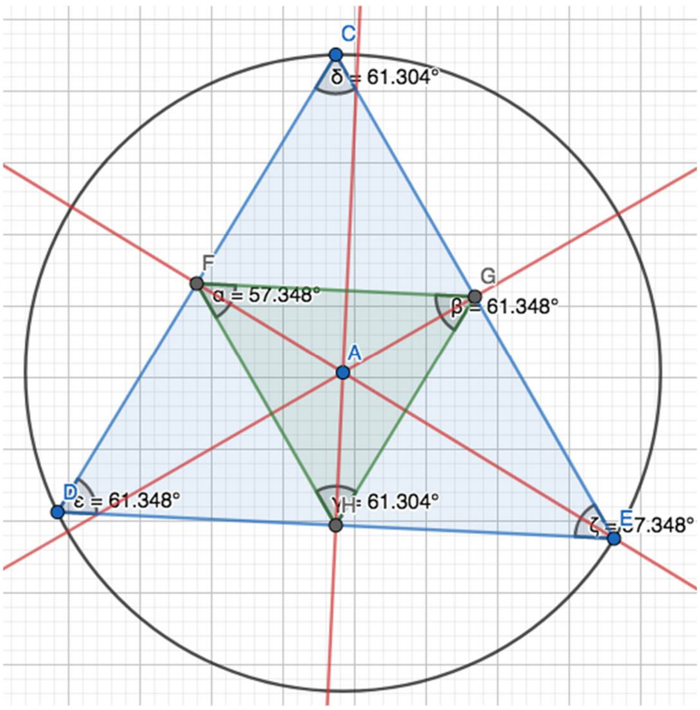
a) Summary

To conclude, by extending the Simson Line theorem, we found the Boomerang theorem. After our researches, we found concrete reasons why the magnitude of the angles formed by the sides of the Boomerang and the sides of the quadrilateral remains the same although the position of the point on the circumference changes and why opposite angles on the circumference are supplementary, and had proven that all of them are not coincidental. We also managed to prove the relationship between the sides and angles. In this process, we used the Thales's theorem. Despite putting in all of our effort, our project is not considered perfect and it needs further extensions.

b) Further Extension

The Simson line may seem to be a simple theory. A triangle in a circle made of the three vertices of the triangle and a point which we can call the Earth Point. Just like how the earth is in the galaxy, revolving around the sun, this point stays in its orbit and goes around the centre of the circle. However, back in the old days, the earth was believed to be the centre of our galaxy. Therefore, instead of putting the earth point at the circumference, the centre of the circle can be put at the centre of the circle. With the same concept of Simson line, following are the modifications of The Simson Line.

- Triangle

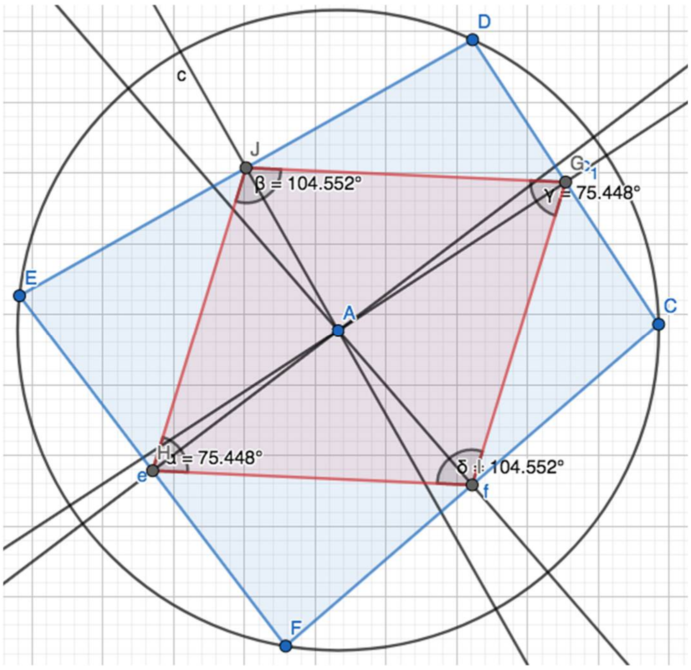


The image shows how a triangle is formed when the nearest three points from the centre of the circle to the sides of the triangles are marked and connected.

∠FGH=∠DCE
∠GHF=∠CDE
∠GFH=∠CED

Three corresponding angles are equal and hence, it can be concluded that triangle FGH is similar is similar to the original triangle CDE. The opposite corresponding angles are equal. Therefore, there are three rhombuses produced: FDGH, CFGH and FGEH. No matter how the three points of the original triangle are displaced, this property will never change.

➢ 4-sided Polygon



When using a 4-sided polygon, a rhombus will be produced as shown in the picture below. Unlike the triangle which produced a smaller version of itself, this polygon produced a rhombus.

However, at this moment, we are unable to explain why and how can it finally leads to these properties. Besides that, the mathematical proof that these properties are correct is still not found. Thus, we kindly encourage those who are interested to continue our journey and unveil what is yet to be found.

VI. References:

http://jwilson.coe.uga.edu/emt668/EMAT6680.2000/Umberger/EMAT6680smu/Assign9smu/pfsimline.html

https://en.m.wikipedia.org/wiki/Simson_line

https://en.m.wikipedia.org/wiki/Quadrilateral/ptolemys-theorem/

https://brilliant.org/wiki
https://en.m.wikipedia.org/wiki/Ptolemy%27s_theorem

http://mathworld.wolfram.com/CyclicQuadrilateral.html

https://www.mathopenref.com/thalestheorem.html

https://en.wikipedia.org/wiki/Thales%27s_theorem