Expansion of Finsler-Hadwiger Theorem

Submitted by S3 PLMGS (S) Students

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A project presented to the Singapore Mathematical Project Festival

2019

Abstract

In mathematics, the Fundamental Theorem of Directly Similar Figures (FTDSF) states that the lines joining two corresponding vertices of similar polygons are divided into equal ratios, then the resulting polygons (which we called the "Constructed Polygon") from these lines are directly similar to the 2 polygons (which we called the "Original Polygon"). Comparably, the Finsler-Hadwiger theorem states that a square (which we called the "Constructed Polygon") can be derived from any two squares (which we called the "Original Polygon") that share a common vertex, by linking the midpoints of the corresponding vertices.

In this project, we came up with the idea of combining the concept of FTDSF and Finsler-Hadwiger theorem, to create more similar figures by using the midpoints of lines that are connected to its corresponding vertices, and the midpoints of lines made by connecting a vertex to all the other vertices of the other polygon.

We used trigonometry to find the ratio of the length of the Constructed Polygon's sides to the length of the Original Polygon's sides, and derived a general formula for this relationship.

We then explored into a different type of case scenario of which the Original Polygons do not share a common vertex, and are apart from each other.

We found out that

(1) Number of constructed polygons is equal to *3n* and the length ratio of the congruent constructed polygon to the original polygon is $\frac{1}{2}$

(2) The ratio of the length of the Constructed Polygon's sides to the length of the Original Polygon's sides, and derived a general formula for this relationship, which is

$$\therefore r_x = \left| \left[\sin\left(\frac{1}{2}\theta + \frac{x-1}{n}\pi \right) \right] \right|$$

where r_x representing the ratio, x representing the number of constructed polygon, n representing number of sides of the polygon, while θ representing the angle between the two sides of the original polygons at their common vertex.

Acknowledgement

The students involved in this report and project thank the school for this opportunity to conduct a research on the Fundamental Theorem of Directly Similar Figures and the Finsler-Hadwiger theorem.

They would like to express their gratitude to the project supervisor, Ms Kok Lai Fong for her guidance in the course of preparing this project.

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Chapter 1: Introduction

1.1 Objectives

In this project, we use different polygons and link different vertices between the Original Polygons to obtain multiple Constructed Polygons in different manners. We want to find out the relationship of the length ratio between the original polygon and the constructed similar polygon.

1.2 Problem

The Fundamental Theorem of Directly Similar Figures only stops at linking two polygons' corresponding vertices to derive a congruent Constructed Polygon. Also, the Finsler–Hadwiger theorem is limited to only a Constructed Square being derived from two Original Squares that share a common vertex. In this project, we use different polygons and link different vertices between the Original Polygons to obtain different Constructed Polygons in different manners.

1.3 Background Information

Paul Finsler, a German and Swiss mathematician, is known for his thesis work concerning *differential geometry*, Finsler spaces, and his contribution to the *foundations of mathematics.* Hugo Hadwiger, a Swiss mathematician, is known for his contribution to geometry, combinatorics, and cryptography. Together, they co-create the Finsler-Hadwiger inequality and Finsler-Hadwiger theorem.

Chapter 2: Literature Review

2.1 Overview

Our proposed problem involves two theorems, namely the Fundamental Theorem of Directly Similar Figures and Finsler-Hadwiger Theorem, which are part of our literature review.

2.2 Fundamental Theorem of Directly Similar Figures

The Fundamental Theorem of Directly Similar Figures says the lines joining two corresponding vertices of similar polygons are divided into equal ratios, then the resulting polygons from these lines are directly similar to the 2 polygons.

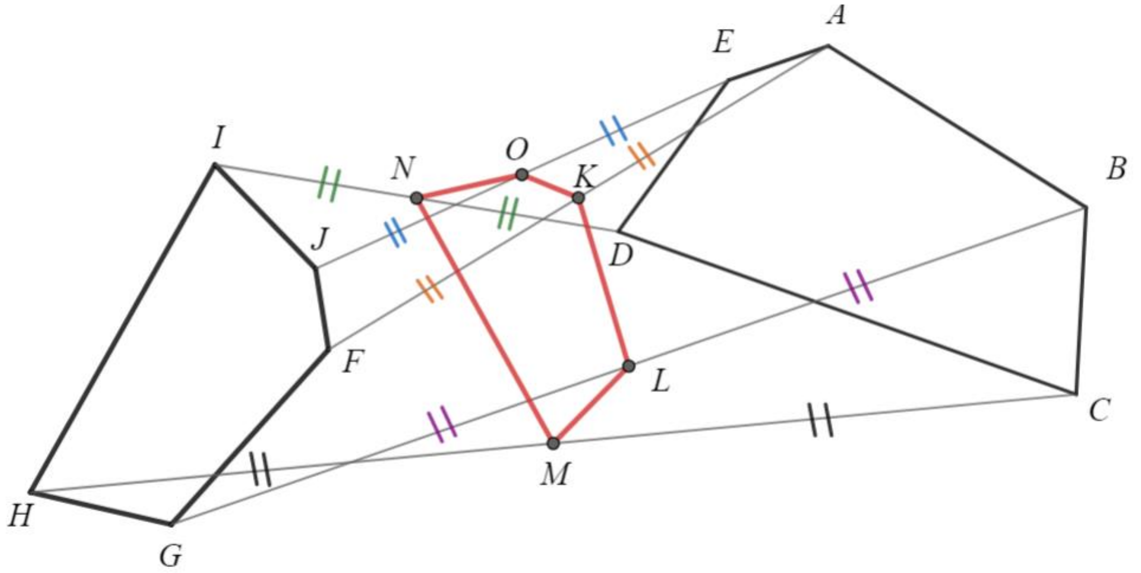

Figure 2.2

For example, in *Figure 2.2* polygon *ABCDE* and polygon *FGHIJ* are similar where point *A* corresponds to point *F*, point *B* to point *G*, point *C* to point *H*, point *D* to point *I*, point *E* to point *J*. Hence, when the lines joining two corresponding vertices are divided into equal ratios, which in this case is 1 to 1, the resulting polygon *KLMNO* is similar to polygon *ABCDE* and polygon *FGHIJ*.

2.3 Finsler-Hadwiger Theorem

The Finsler-Hadwiger Theorem states that a square can be derived from two squares that share a vertex. The new square will have the centre of square 1, centre of square 2, and the midpoints of the linking segments as its vertices.

From *Figure 2.3*, two squares, square *ABCD* and square *AEGF* share a common vertex which is *A*. Hence, using the Finsler-Hadwiger Theorem, a square can be derived from the two squares. The vertices of the new square is centre *J*, midpoint of *BE* is *H*, centre *K* and midpoint of *DF* is *I*. The new square is square *KHJI*.

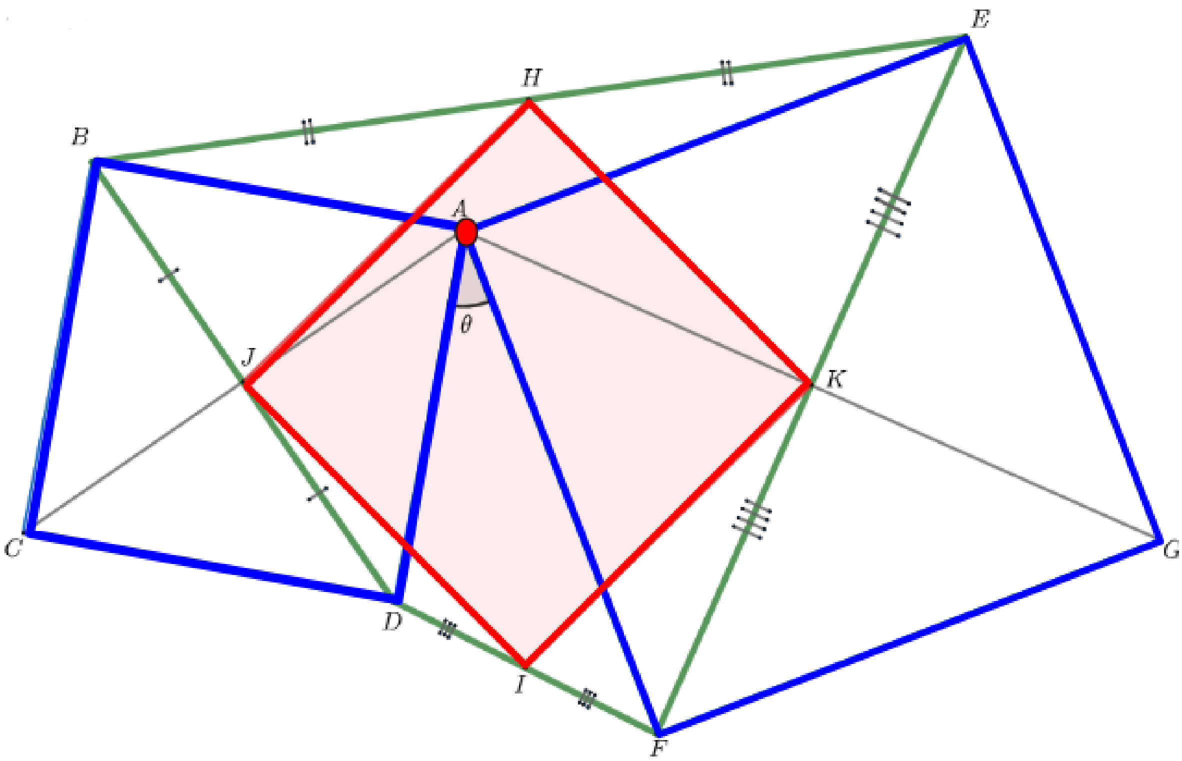

Figure 2.3

The Fundamental Theorem of Directly Similar Figures states that the resulting lines from the points that are divided into equal ratios will form a similar polygon.

Square *ABCD* and square *AEGF* are the parent squares. Using the same ratio of 1 : 1 for all the corresponding sides - *AG, BE, CA, AF* - the points taken using the ratio 1 : 1 will be point *K, H, J, I*.

By the Fundamental Theorem of Directly Similar Figures, points *KHJI* when are joint by lines, will form a similar square to its parents, square *ABCD* and square *AEGF*.

Chapter 3: Methodology

3.1 Overview

We are going to construct possible similar polygons that are derived from two congruent regular polygons which share a common vertex by applying the Fundamental Theorem of Directly Similar Figures and the Finsler-Hadwiger Theorem. After that, we are going to research more on the relation between the Constructed Polygons and the Original Polygon. For illustration purposes we are using pentagon, however our explanation will refer to *n*-sided polygon instead of the illustrated pentagon, unless otherwise stated.

3.2 Possible Constructed Polygons

Possible constructed polygon that are similar to original regular polygon can be constructed when all the vertices of one of the original polygon correspond to only one vertex of the other polygon.

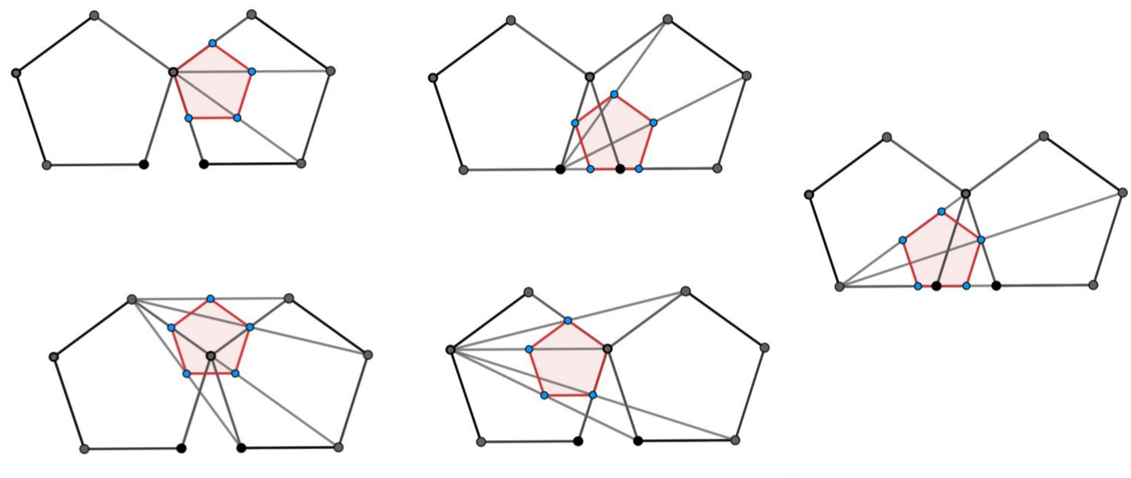

Figure 3.2 (1)

In *Figure 3.2 (1)* all of the vertices of the polygon on the right correspond to one of the vertices of the polygon on the left. By taking the midpoint of the lines that connect the vertices of the polygon on the right to one of the vertices of the polygon on the left, a similar polygon can be constructed. In *Figure 3.2 (1)* the constructed polygons are represented by the red polygons. Since each *n*-sided polygon has *n* number of vertices, the number of possible polygons that can be constructed using this method is *n*.

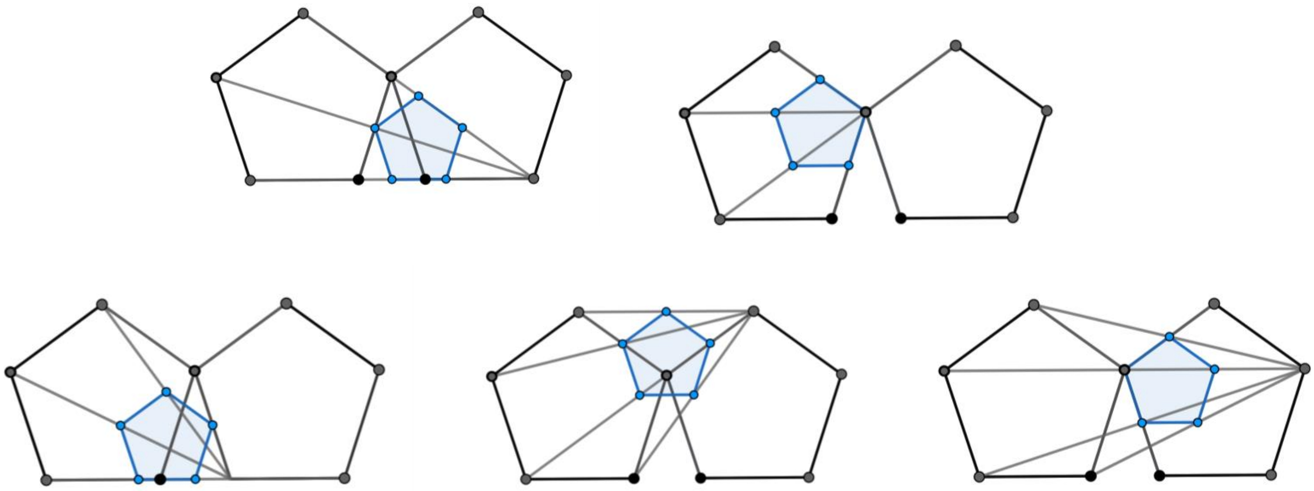

Figure 3.2 (2)

In *Figure 3.2 (2)* all of the vertices of the polygon on the left correspond to one of the vertices of the polygon on the right. In *Figure 3.2 (2)* the constructed polygons are represented by the blue polygons. Since each n-sided polygon has n number of vertices, the number of possible polygons that can be constructed using this method is n.

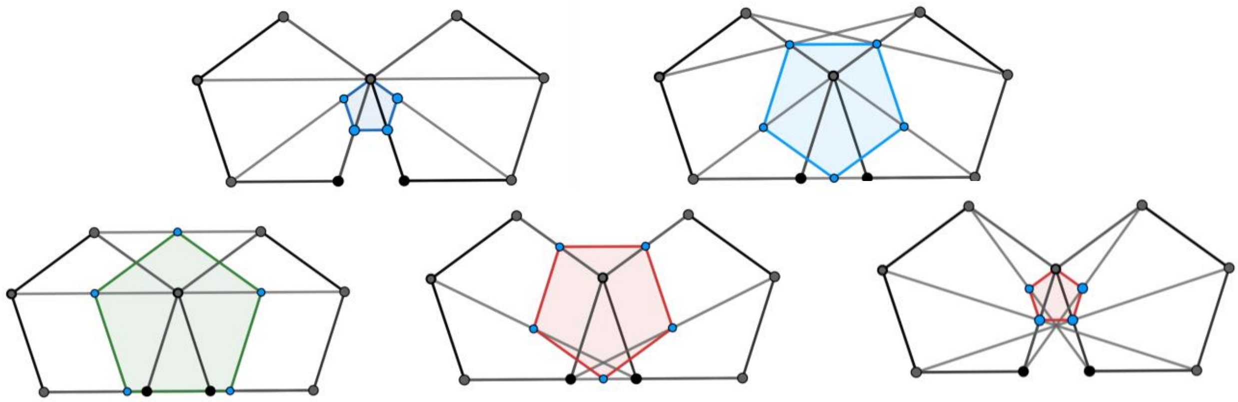

Figure 3.2 (3)

In *Figure 3.2 (3)* each vertex of one polygon correspond to each vertices of the other polygon. Since each vertex have n number of possible vertices to correspond with for n-sided polygon, the number of possible polygons that can be constructed using this method is n.

From *Figure 3.2 (1), Figure 3.2 (2),* and *Figure 3.2 (3)* total number of possible constructed polygon is $n + n + n = 3n$.

3.3 Length Ratio of Congruent Constructed Polygons to the Original Polygons

There are *3n* number of possible constructed polygons and upon further research, there are total of *2n* number of polygons that are congruent to each other.

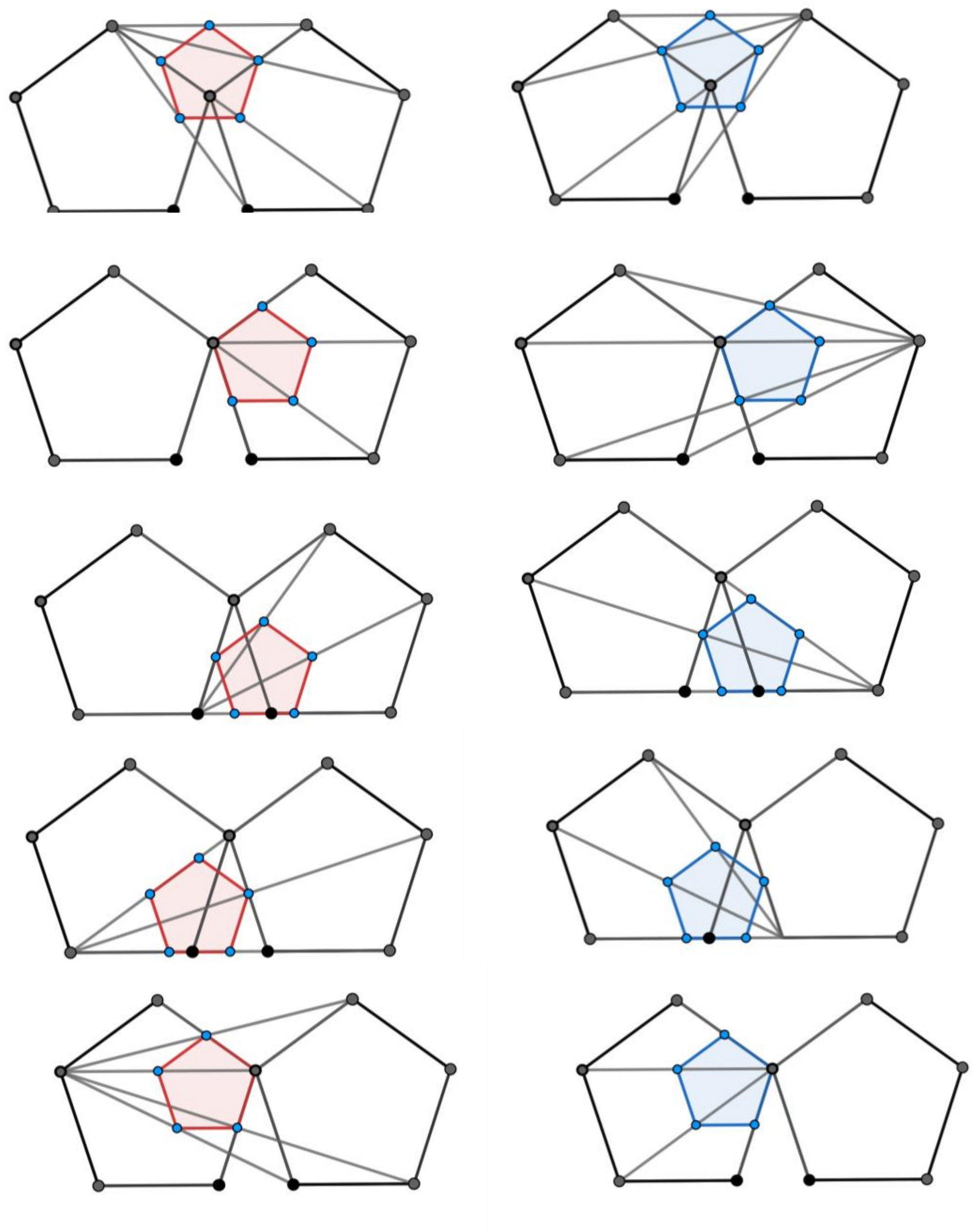

Figure 3.3 (1)

The constructed polygons in the *Figure 3.3 (1)* are example of constructed polygons are congruent. The proof of them being congruent to each other is the length ratio of the congruent constructed polygon to the original polygons.

To take one example of congruent constructed polygons,

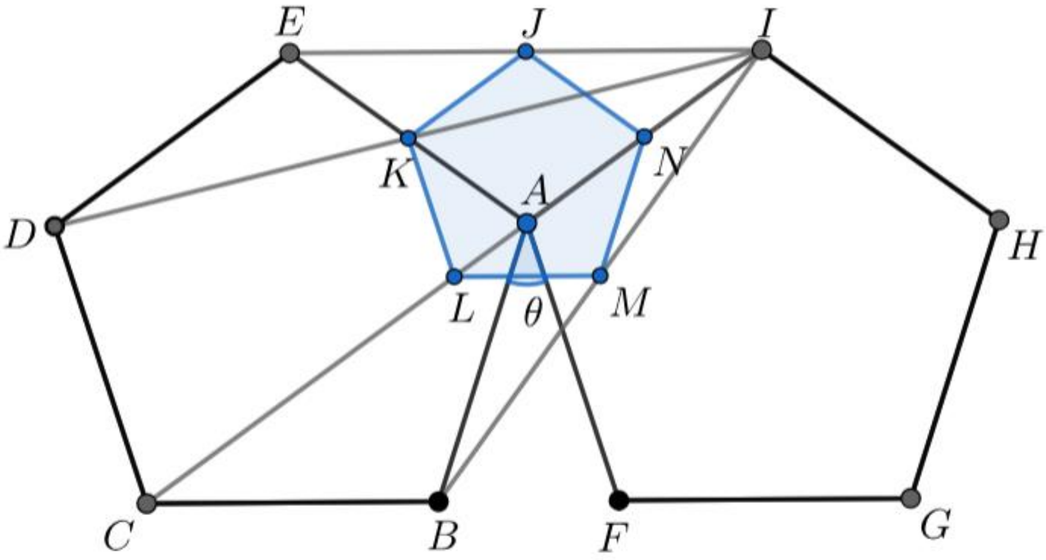

Figure 3.3 (2)

The congruent constructed polygons are polygons with vertices that lie on the midpoint of lines connecting every vertex of one polygon to only one vertex of the other polygon. In the figure above, the child pentagon's vertices are the midpoint of line IA, IB, IC, ID, and IE.

Using Triangle Midpoint Theorem,
The line segment connecting the midpoints of two sides of a triangle is parallel to the third side and is congruent to one half of the third side.

Since *M* is the midpoint of *IB* and *L* is the midpoint of IC,

$$LM \mathbin{//} CB$$

$$LM = \frac{1}{2}CB$$

$$\therefore\ LM : CB = 1:2$$

Since the *2n* number of polygons would have the same length ratio of 1 to 2 with the length of the original polygons and they are similar, they are congruent to each other.

3.4 Length Ratio of Constructed Polygons that are not Congruent to the Original Polygons

Since *2n* number of constructed polygons are congruent to each other, *n* number of constructed polygons are not congruent to any other constructed polygons. To derive the length ratio of constructed polygons that are not congruent, we are using pentagons as examples.

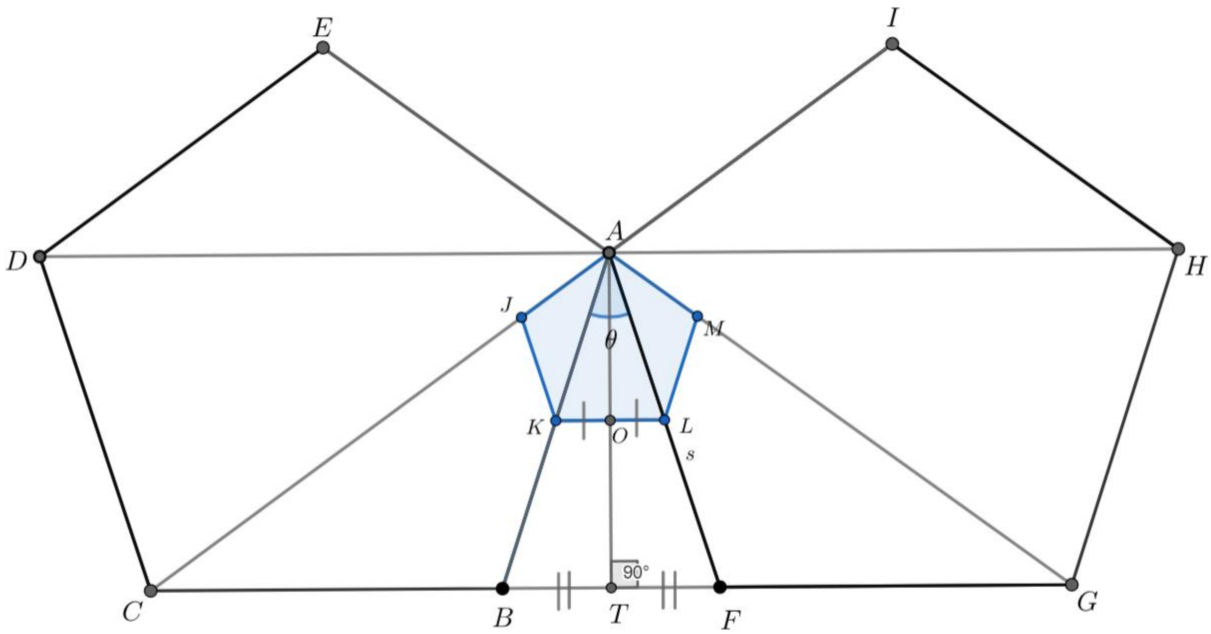

Figure 3.4 (1)

The blue pentagon in *Figure 3.4 (1)* depicts the first constructed polygon that are not congruent. From here onwards we are referring the constructed polygons that are not congruent as $poly_1$, $poly_2$, $poly_3$, …., $poly_n$. In *Figure 3.4 (1)* $poly_1$ is represented by the blue pentagon.

For the $poly_1$, the vertices of the pentagon is the midpoint of line *AF*, *AB*, *IC*, *DH*, and *EG*. If a perpendicular line is extended from point *A* to the midpoint of the line *BF*, *T*, the line *AT* will bisect *KL*, the side of the $poly_1$, and *BF*.

$$\because \Delta AOL \text{ is similar to } \Delta ATF,$$

$$\frac{OL}{TF} = \frac{AL}{AF}$$

$\because$ *L* is the midpoint of *AF*,

$$\frac{OL}{TF} = \frac{\frac{1}{2}AF}{AF}$$

$$\frac{OL}{TF} = \frac{1}{2} \qquad \text{.....................(1)}$$

$\because$ *AT* bisects $\angle BAF$,

$$\angle TAF = \frac{1}{2}\theta$$

$\because$ $\angle$ATF is a right angle,

$$\sin\left(\frac{1}{2}\theta\right) = \frac{TF}{AF}$$

$$TF = AF\sin\left(\frac{1}{2}\theta\right) \qquad \text{..................... (2)}$$

Sub equation (2) to (1),

$$\frac{OL}{AF\sin\left(\frac{1}{2}\theta\right)} = \frac{1}{2}$$

$$OL = \frac{AF \sin\left(\frac{1}{2}\theta\right)}{2}$$

$$\therefore KL = 2OL$$

$$KL = 2\left[\frac{AF \sin\left(\frac{1}{2}\theta\right)}{2}\right]$$
$$KL = AF \sin\left(\frac{1}{2}\theta\right)$$
$$\frac{KL}{AF} = \sin\left(\frac{1}{2}\theta\right)$$

Therefore, length ratio of the sides of poly$_1$ to the sides of the original polygon is $\boldsymbol{sin\left(\frac{1}{2}\theta\right)}$

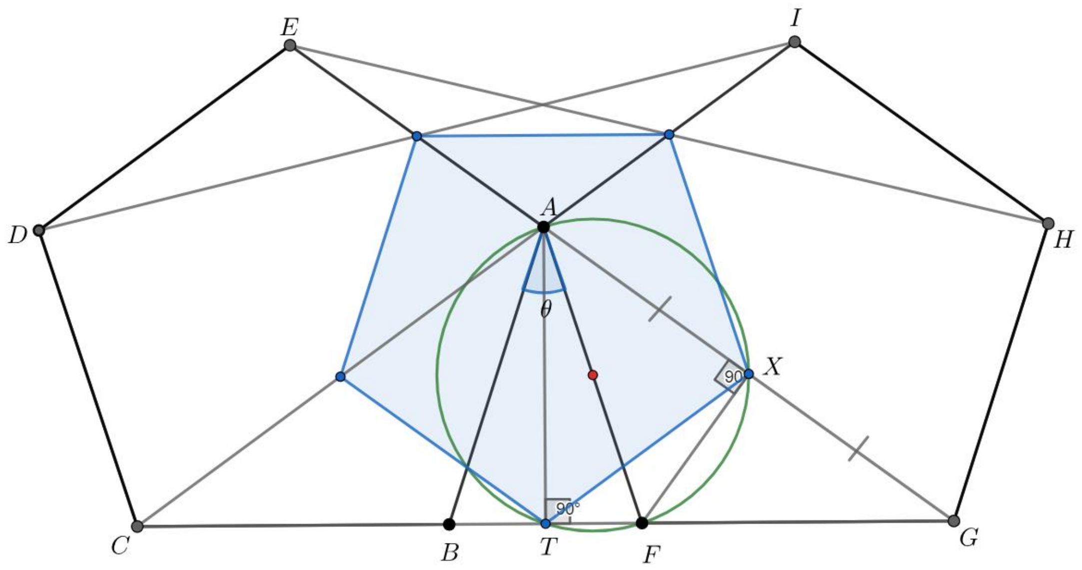

Figure 3.4 (2)

The blue pentagon in *Figure 3.4 (2)* depicts the second constructed polygon that are not congruent. In *Figure 3.4 (1) poly$_2$* is represented by the blue pentagon.

For the *poly$_2$*, the vertices of the pentagon is the midpoint of line *ID*, *AC*, *BF*, *AG*, and *EH*.

As *X* is the midpoint of *AG*, and *AF* = *FG*,

$$FX \perp AG$$

As *T* is the midpoint of *BF*, and *AB* = *AF*,

$$AT \perp BF$$

Hence,

$$\angle ATF = \angle AXF = \frac{\pi}{2}$$

$$\angle ATX = \angle AFX$$ (angles in the same segment)

$\angle AFG$ is one interior angle of a pentagon.

$$\therefore \angle AFG = \frac{(5-2)\pi}{5}$$ (int angles of polygon)

$$= \frac{3}{5}\pi$$

$$\therefore \angle FAG = \frac{\pi - \frac{3}{5}\pi}{2}$$ (base angles of isosceles triangle)

$$= \frac{1}{5}\pi$$

Using trigonometry,

$$\frac{AX}{AF} = \sin \angle AFX$$

$$AF = \frac{AX}{\sin \angle AFX}$$ ………………… (3)

Using the sine rule on ΔATF,

$$\frac{AX}{\sin \angle AFX} = \frac{XT}{\sin(\angle TAF + \angle FAX)}$$

$$\frac{AX}{\sin \angle AFX} = \frac{XT}{\sin\left(\frac{1}{2}\theta + \frac{1}{5}\pi\right)}$$ ………………… (4)

Sub (3) into (4)

$$\frac{XT}{\sin\left(\frac{1}{2}\theta + \frac{1}{5}\pi\right)} = AF$$
$$\frac{XT}{AF} = \sin\left(\frac{1}{2}\theta + \frac{1}{5}\pi\right)$$

Therefore, length ratio of the sides of *poly*$_2$ to the sides of the original polygon is

$\boldsymbol{sin\ \left(\frac{1}{2}\theta + \frac{1}{5}\pi\right)}$**.**

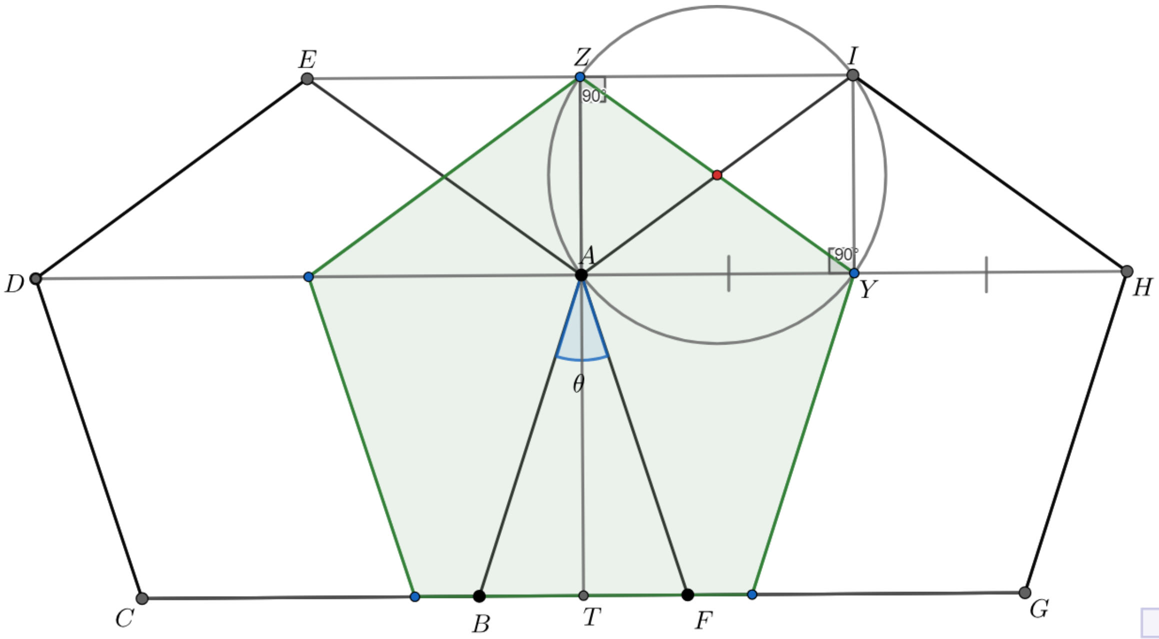

Figure 3.4 (3)

The green pentagon in *Figure 3.4 (3)* depicts the third constructed polygon that are not congruent. In *Figure 3.4 (3) poly*$_3$ is represented by the green pentagon.

For the *poly*$_3$, the vertices of the pentagon is the midpoint of line *IE, AD, FC, BG,* and *AH.*

As *Y* is the midpoint of *AH* and *AI* = *IH*,

$$IY \perp AH$$

As Z is the midpoint of *EI* and *AE* = *AI*,

$$AZ \perp IE$$

$$\therefore \angle AZI = \angle AYI = \frac{\pi}{2}$$

$$\angle ZYA = \angle ZIA$$ (angles in same segment)

$\angle AIH$ is one interior angle of pentagon

$$\angle AIH = \frac{(5-2)\pi}{5}$$

$$= \frac{3}{5}\pi$$

$$\therefore \angle IAH = \frac{\pi - \frac{3}{5}\pi}{2} \quad = \frac{1}{5}\pi$$

As *AZ*?*IE* and *AT*?*BF* while *Z* is the midpoint of *IE* and *T* is the midpoint of *BF*, points *T*, *A*, *Z* are collinear.

$$\angle AIH = \angle IAF$$ (int angles of polygon)

$$\angle ZAH = \pi - \angle TAH$$ (supplementary angles)

$$\angle TAH = \frac{1}{2}\theta + \angle IAF - \angle IAH$$

$$= \frac{1}{2}\theta + \frac{3}{5}\pi - \frac{1}{5}\pi$$

$$= \frac{1}{2}\theta + \frac{2}{5}\pi \quad \angle ZAH = \pi - \left(\frac{1}{2}\theta + \frac{2}{5}\pi\right)$$

Using trigonometry,

$$\sin\angle ZIA = \frac{ZA}{AI}$$

$$AI = \frac{ZA}{\sin\angle ZIA} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Using the sine rule on $\triangle AZI$,

$$\frac{ZI}{\sin\angle ZAI} = \frac{AI}{\sin\angle AZI} = \frac{ZA}{\sin\angle ZIA}$$

$$\frac{ZA}{\sin \angle ZIA} = \frac{ZY}{\sin \angle ZAH} \qquad \text{.................. (6)}$$

Sub (5) into (6)

$$\frac{ZY}{\sin \angle ZAH} = AI$$

$$\frac{ZY}{AI} = \sin\left[\pi - \left(\frac{1}{2}\theta + \frac{2}{5}\pi\right)\right]$$

$$\frac{ZY}{AI} = \sin\left(\frac{1}{2}\theta + \frac{2}{5}\pi\right)$$

Therefore, length ratio of the sides of *poly*$_3$ to the sides of the original polygon is

$\boldsymbol{sin\ \left(\frac{1}{2}\theta + \frac{2}{5}\pi\right)}$.

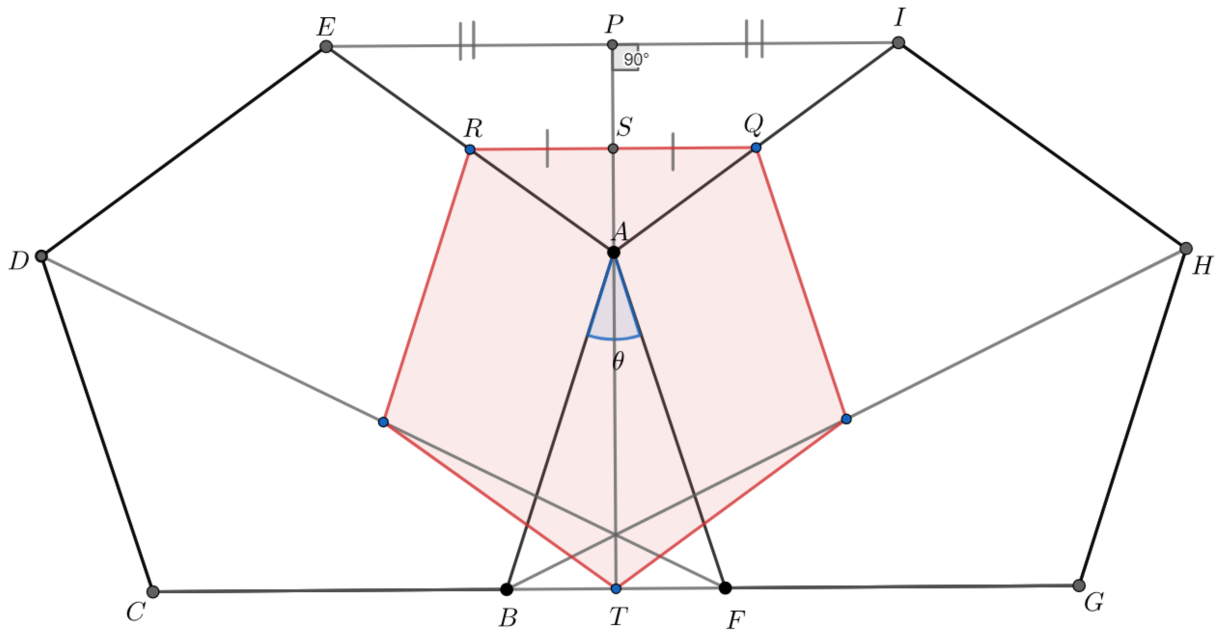

Figure 3.4 (4)

The red pentagon in *Figure 3.4 (4)* depicts the fourth constructed polygon that are not congruent. In *Figure 3.4 (4) poly*$_4$ is represented by the red pentagon.

For the poly$_4$, the vertices of the pentagon is the midpoint of line *AI*, *AE*, *DF*, *CG* and *BH*.

As Q is the midpoint of AI,

$$AQ = \frac{1}{2}AI$$

Then $\triangle ASQ$ is similar to $\triangle API$,

$$\frac{SQ}{PI} = \frac{AQ}{AI}$$
$$\frac{SQ}{PI} = \frac{\frac{1}{2}AI}{AI}$$
$$\frac{SQ}{PI} = \frac{1}{2} \qquad \text{………………… (7)}$$

$\therefore \triangle API$ is a right-angled triangle,

$$\sin \angle PAI = \frac{PI}{AI}$$
$$AI \sin \angle PAI = PI \qquad \text{………………… (8)}$$

Sub (8) to (7)

$$\frac{SQ}{AI \sin \angle PAI} = \frac{1}{2}$$
$$SQ = \frac{AI \sin \angle PAI}{2}$$
$$2SQ = 2\left[\frac{AI \sin \angle PAI}{2}\right]$$
$$RQ = AI \sin \angle PAI$$
$$\frac{RQ}{AI} = \sin \angle PAI$$

Since P, A, and T are collinear

$$\angle PAI = \pi - \angle TAI \text{ (supplementary angles)}$$
$$\angle PAI = \pi - (\angle TAF + \angle IAF)$$

$$\angle PAI = \pi - \left(\frac{1}{2}\theta + \frac{3}{5}\pi\right)$$
$$\frac{RQ}{AI} = \sin\left[\pi - \left(\frac{1}{2}\theta + \frac{3}{5}\pi\right)\right]$$
$$\frac{RQ}{AI} = \sin\left(\frac{1}{2}\theta + \frac{3}{5}\pi\right)$$

Therefore, length ratio of the sides of $poly_4$ to the sides of the original polygon is
$\boldsymbol{sin\ \left(\frac{1}{2}\theta + \frac{3}{5}\pi\right)}$.

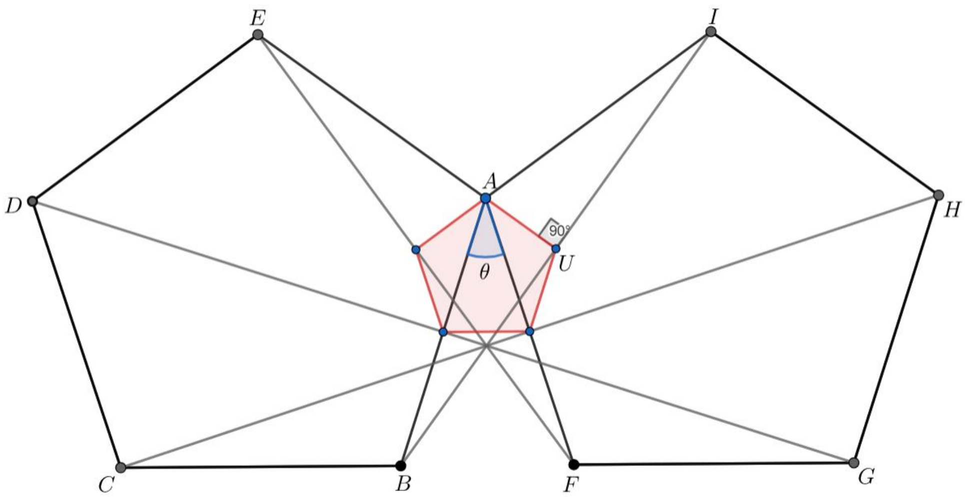

Figure 3.4 (5)

The red pentagon in *Figure 3.4 (5)* depicts the fifth constructed polygon that are not congruent. In *Figure 3.4 (5)* $poly_5$ is represented by the red pentagon.

For the $poly_5$, the vertices of the pentagon is the midpoint of line *BI, CH, DG, EF* and point *A*.

$$\therefore AB = AI$$

$\triangle ABI$ is an isosceles triangle

$$\angle FAI = \frac{(5-2)\pi}{5} \quad \text{(int angles of polygon)}$$

$$\angle FAI = \frac{3}{5}\pi$$
$$\angle BAI = \angle BAF + \angle FAI$$
$$\angle BAI = \theta + \frac{3}{5}\pi$$
$$\angle AIB = \frac{\pi - \left(\theta + \frac{3}{5}\pi\right)}{2}$$ (base angles of isosceles triangle)
$$\angle AIB = \frac{1}{2}\pi - \frac{1}{2}\theta - \frac{3}{10}\pi$$
$$\angle AIB = \frac{1}{5}\pi - \frac{1}{2}\theta$$

Since *U* is the midpoint of *BI* and *AB* = *AI*

AU ⍰ *BI*

Thus, $\triangle AIU$ is a right-angled triangle

$$\sin\angle AIU = \sin\left(\frac{1}{5}\pi - \frac{1}{2}\theta\right)$$
$$\sin\angle AIU = \sin\left[\pi - \left(\frac{1}{5}\pi - \frac{1}{2}\theta\right)\right]$$
$$\sin\angle AIU = \sin\left(\pi + \frac{1}{2}\theta - \frac{1}{5}\pi\right)$$
$$\sin\angle AIU = \sin\left(\frac{1}{2}\theta + \frac{4}{5}\pi\right)$$
$$\sin\angle AIU = \frac{AU}{AI}$$
$$\frac{AU}{AI} = \sin\left(\frac{1}{2}\theta + \frac{4}{5}\pi\right)$$

Therefore, length ratio of the sides of $poly_5$ to the sides of the original polygon is

$\boldsymbol{sin\ \left(\frac{1}{2}\theta + \frac{4}{5}\pi\right)}$.

Using the example of a pentagon,
The ratio of the length of non-congruent child's side to the length of parent's side is (represented by $r_1, r_2, r_3, \ldots, r_n$)

For poly$_1$, $r_1 = sin\left(\frac{1}{2}\theta\right)$

For poly$_2$, $r_2 = sin\left(\frac{1}{2}\theta + \frac{1}{5}\pi\right)$

For poly$_3$, $r_3 = sin\left(\frac{1}{2}\theta + \frac{2}{5}\pi\right)$

For poly$_4$, $r_4 = sin\left(\frac{1}{2}\theta + \frac{3}{5}\pi\right)$

For poly$_5$, $r_5 = sin\left(\frac{1}{2}\theta + \frac{4}{5}\pi\right)$

From the derived formula above, if a general formula is to be made, the general formula of the ratio is

$$\therefore r_x = \left|\left[\sin\left(\frac{1}{2}\theta + \frac{x-1}{n}\pi\right)\right]\right|$$

where r_x representing the ratio, x representing the number of constructed polygon, and n representing number of sides of the polygon.

Chapter 4: Conclusion

4.1 Conclusion

In mathematics, the Fundamental Theorem of Directly Similar Figures states that the lines joining two corresponding vertices of similar polygons are divided into equal ratios, then the resulting polygons (which we called the “Constructed Polygon”) from these lines are directly similar to the 2 polygons (which we called the “Original Polygon”). Comparably, the Finsler-Hadwiger theorem states that a square (which we called the “Constructed Polygon”) can be derived from any two squares (which we called the “Original Polygon”) that share a common vertex, by linking the midpoints of the corresponding vertices.
In this project, we came up with the idea of combining the concept of Fundamental Theorem of Directly Similar Figures and Finsler-Hadwiger theorem, to create more similar figures by using the midpoints of lines that are connected to its corresponding vertices, and the midpoints of lines made by connecting a vertex to all the other vertices of the other polygon.

We found out a few facts, including:
Number of constructed polygons is equal to *3n* and the length ratio of the congruent constructed polygon to the original polygon is $\frac{1}{2}$

We used trigonometry to find the ratio of the length of the Constructed Polygon’s sides to the length of the Original Polygon’s sides, and derived a general formula for this relationship, which is

$$\therefore r_x = \left| \left[\sin\left(\frac{1}{2}\theta + \frac{x-1}{n}\pi \right) \right] \right|$$

where r_x representing the ratio, x representing the number of constructed polygon, n representing number of sides of the polygon, while θ representing the angle between the two sides of the original polygons at their common vertex.

Reference:

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