Application of Nine Point Circle Theorem

Submitted by S3 PLMGS(S) Students

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A project presented to the Singapore Mathematical Project Festival

2019

Abstract

In mathematics geometry, a nine-point circle is a circle that can be constructed from any given triangle, which passes through nine significant concyclic points defined from the triangle. These nine points come from the midpoint of each side of the triangle, the foot of each altitude, and the midpoint of the line segment from each vertex of the triangle to the orthocentre, the point where the three altitudes intersect.

In this project we carried out last year, we tried to construct nine-point circles from triangulated areas of an *n*-sided polygon (which we call the "Original Polygon) and create another polygon by connecting the centres of the nine-point circle (which we call the "Image Polygon"). From this, we were able to find the area ratio between the areas of the original polygon and the image polygon. Two equations were found after we collected area ratios from various *n*-sided regular and irregular polygons.

Acknowledgement

The students involved in this project would like to thank the school for the opportunity to participate in this competition.

They would like to express their gratitude to the project supervisor, Ms Kok Lai Fong for her guidance in the course of preparing this project.

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Chapter 1 - Introduction

1.1 Objective

The aim of our project is to find the relationship between the area of an n-sided polygon to its image polygon, that is formed by dividing the polygon into n-triangles and connecting all the centres of the nine-point circles of each triangle. Each triangle is formed by constructing lines from the centre of the polygon to its vertices.

1.2 Insight

Inspired by the Homothetic Transformation and a diagram of A Triangle Formed by the Centers of Three Nine-Point Circles, we used the centers of the nine-point circles of each triangle in a polygon (which we called the “original polygon”) to form another polygon (which we called the “image polygon”).

1.3 Background

The nine-point circle owes its discovery to a group of famous mathematicians over the course of about 40 years, though it is most generally attributed to Karl Feuerbach, a German mathematician who "rediscovered" it in the nineteenth century (however it was known even to Euler). In 1765 Euler proved that the circle through the three feet of the altitudes also passed through the midpoints of the sides. Sometime later, Feuerbach showed that the three midpoints of the segments from the vertices to the orthocenter were on the same circle.

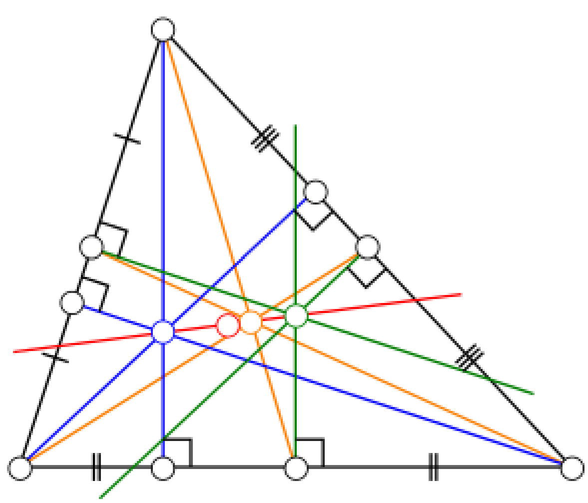
Chapter 2 - Literature Review

2.1 Overview

In order to further extend and apply the Nine-Point Circle Theorem to *n*-sided polygons, we would need some prior knowledge. Here, we have sectioned our Literature Review into 4 parts; the Euler Line, The Nine Point Circle, Homothetic Transformation, and a diagram A Triangle Formed by the Centers of Three Nine-Point Circles.

2.2 The Euler Line

The Euler line, or the Euler segment, consists of the orthocenter, centroid, and circumcenter. The orthocenter of a triangle is the point of concurrency of its three altitudes, which are the perpendicular segments from a vertex to the line of the opposite side. The centroid of a triangle is the point of concurrency of the three medians, which are the segments from a vertex to the midpoint of the opposite side. The circumcenter is the point of concurrency of the three bisectors of the sides of the triangle and is thus, equidistant from the three vertices of the triangle.



2.2 The Nine Point Circle

In geometry, the **nine-point circle** is a circle that can be constructed for any given triangle. It is so named because it passes through nine significant concyclic points defined from the triangle. These nine points are: the midpoint of each side of the triangle (Ma, Mb, Mc), the foot of each altitude (Ha, Hb, Hc), and the midpoint of the line segment from each vertex of the triangle to the orthocenter(H) (Ea, Eb, Ec), (where the three altitudes meet; these line segments lie on their respective altitudes).

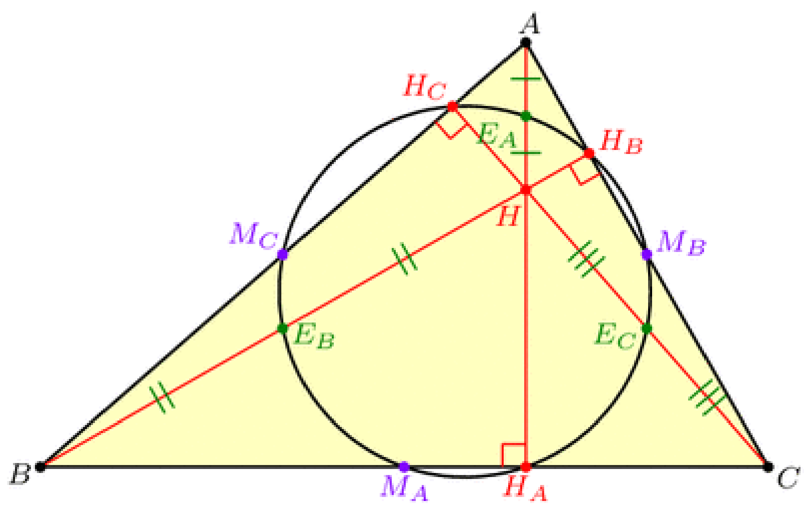


figure 2.2.1

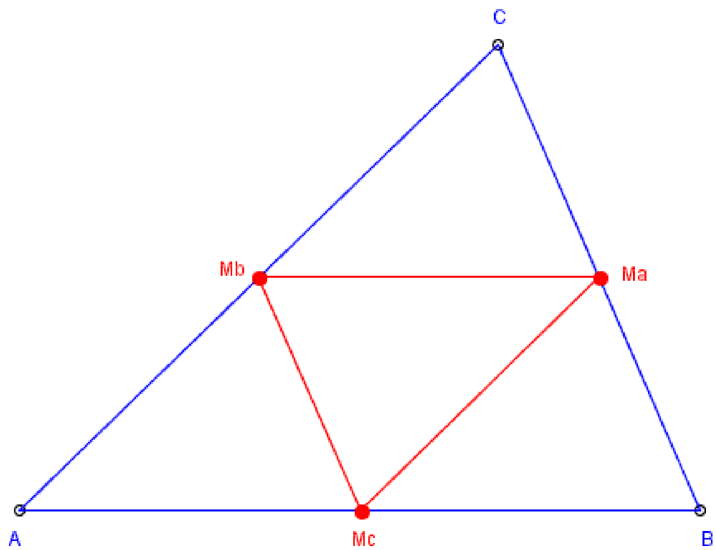


figure 2.2.2

As shown in *figure 2.2.2* above, the medial triangle of a triangle is the triangle with vertices at the midpoints of the triangle's sides.

As shown in *figure 2.2.3* below, the Orthic triangle of a triangle is the triangle with vertices at the feet of the altitudes.

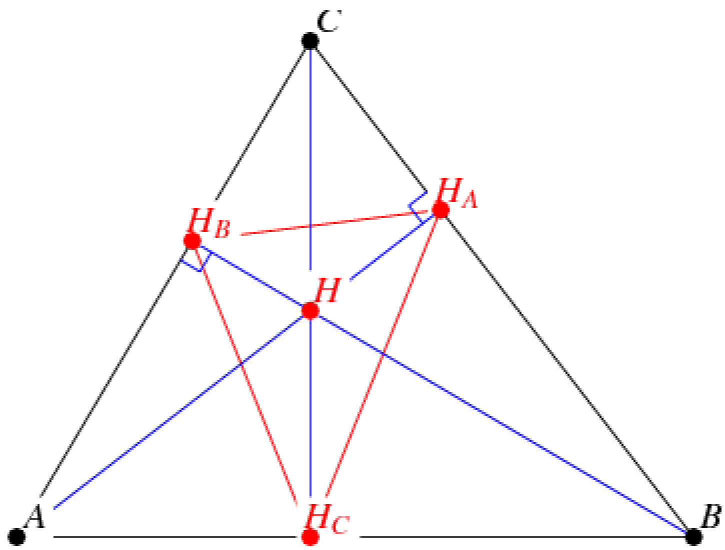


figure 2.2.3

By doing a dilation of one-half using the orthocenter as the center of dilation, we can construct an image triangle whose vertices are the midpoints of the segments from the vertices to the orthocenter, as shown in *figure 2.2.4* below.

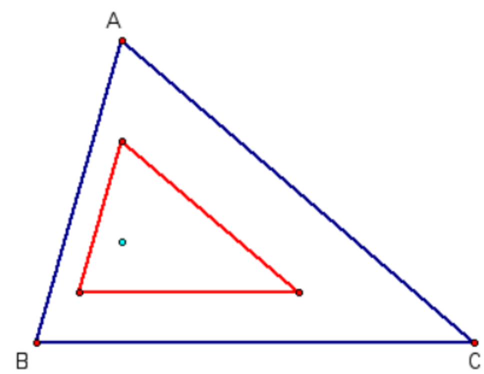


Figure 2.2.4

Now, let's compare the Euler segments of each sub-triangle by observing the medial triangle, orthic triangle, and the image triangle in the same construction.

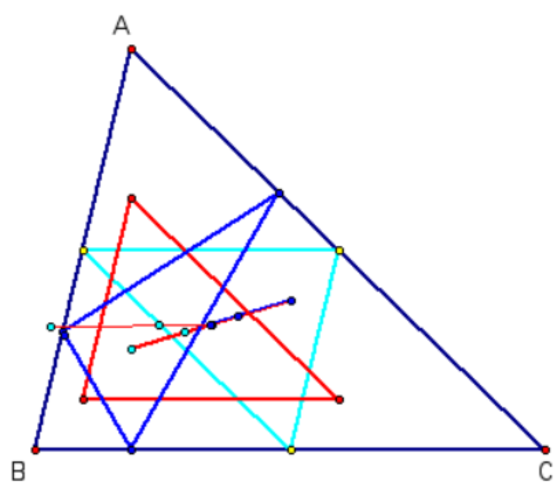


Figure 2.2.5

These three unique sub-triangles all share the same circumcenter. Thus, the conclusion is that there is a circle that is the common circumcircle for these three special sub-triangles. This circle is in fact, the nine point circle.

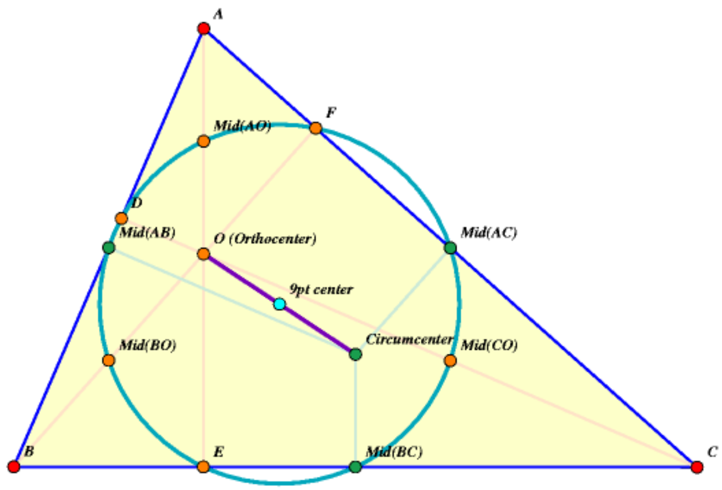


Figure 2.2.6

2.3 Homothetic Transformation

In mathematics, homothetic transformation is a transformation of an affine space with respect to a fixed homothetic centre of point S and a nonzero number of λ as its ratio which sends any point M to another point N such that the segment SM lies on the same line as SN, but scaled by a factor λ.

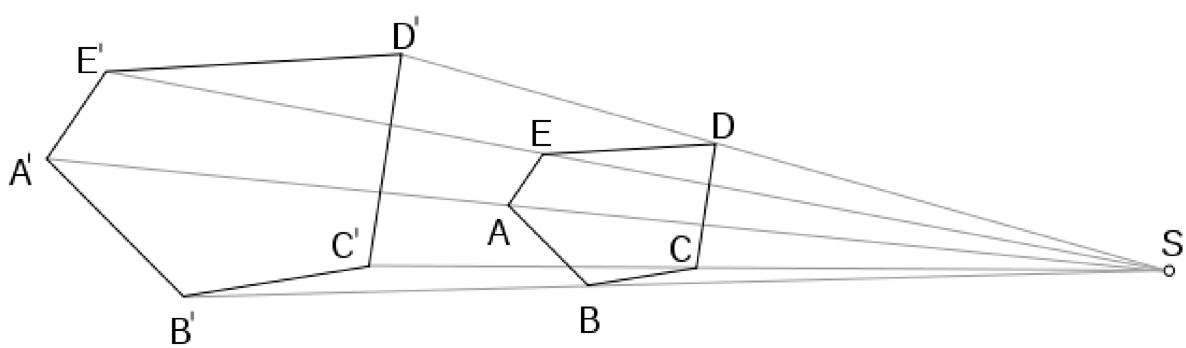


figure 2.3.1

In Euclidean geometry, a mathematical system attributed to Alexandrian Greek mathematician Euclid, homotheties are the similarities that fix a point and either preserve or reserve the direction of all vectors, depending on the value of the λ whether it is positive or negative. Together with translations, or geometric transformations that moves every point of a figure or a space by the same distance in a given direction, all homotheties of an affine or Euclidean space form a group of dilations.

Here are some examples of homothetic transformation

$H(A,2)$ is applied to triangle ABC:

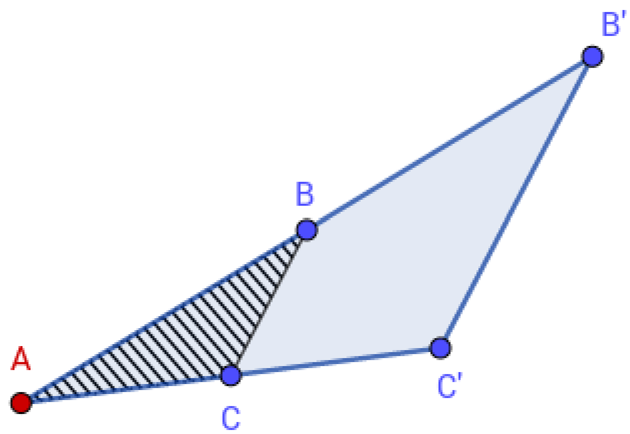


figure 2.3.2

$H(T, ½)$ applied to a circle ⌈’ , where T is a point on the circumference :

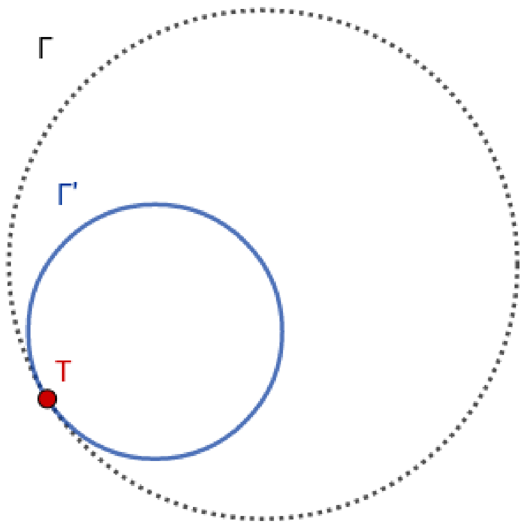


figure 2.3.3

2.4 A Triangle Formed by the Centers of Three Nine-Point Circles

Let ABC be a triangle and let A', B', and C' be the midpoints of BC, AC, and AB, respectively. Let A'', B'', and C'' be the centers of the nine-point circles of the triangles $AB'C'$, $BC'A'$, and $CA'B'$, respectively. Then $A''B''C''$ is homothetic with ABC in the ratio 1:2.

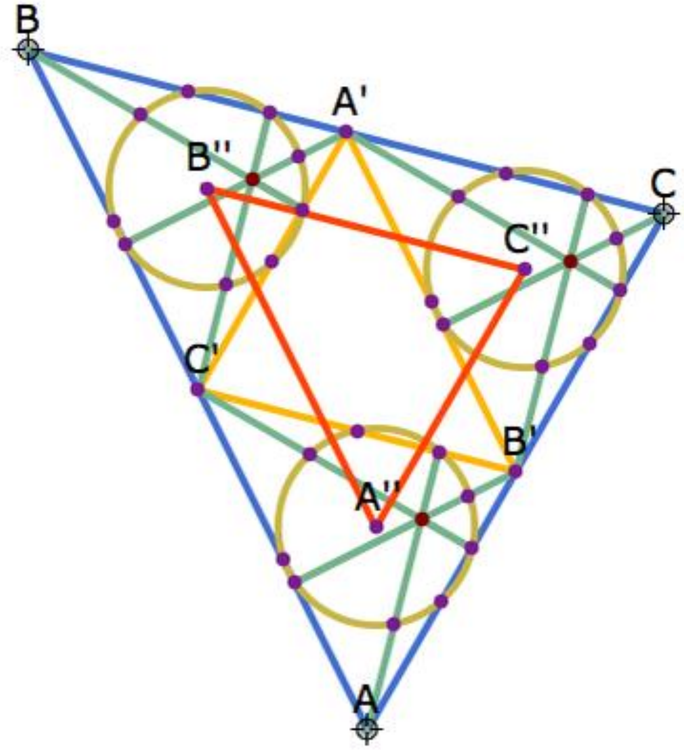


figure 2.4.1

Chapter 3 - Methodology

3.1 Overview

Firstly, we are going to divide an *n*-sided polygon into *n* number of triangles by constructing lines from the centre of the polygon to the vertices of the polygon. For illustration, we will only use regular *n*-sided polygons in this chapter. After dividing the *n*-sided polygon into *n* number of triangles, we can then construct the nine-point circle of each triangle. The centres of these nine-point circles will act as the vertices of an image polygon that we can form by constructing lines connecting the centres of these nine-point circles. The image polygon will be similar to the original polygon in the case of regular n-sided polygons. From this, we are able to find the relationship between the area of the original polygon and the image polygon formed in terms of ratio.

3.2 Finding The Center of The Polygon

For even *n*-sided polygons, construct lines connecting each vertex to its opposite vertex. The centre of the polygon would be the point where all the lines connecting the vertices intersect, as shown in *figure 3.2.1* below.

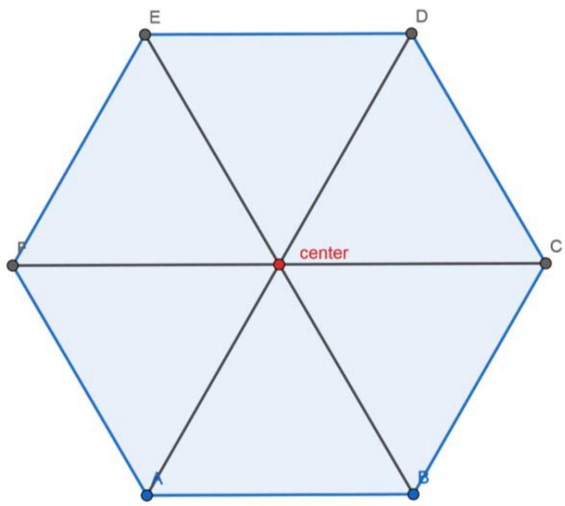


figure 3.2.1

For odd *n*-sided polygons, construct perpendicular lines from each vertex to its opposite side. The centre of the original polygon would also be the point where all the perpendicular lines meet, as shown in *figure 3.2.2* below.

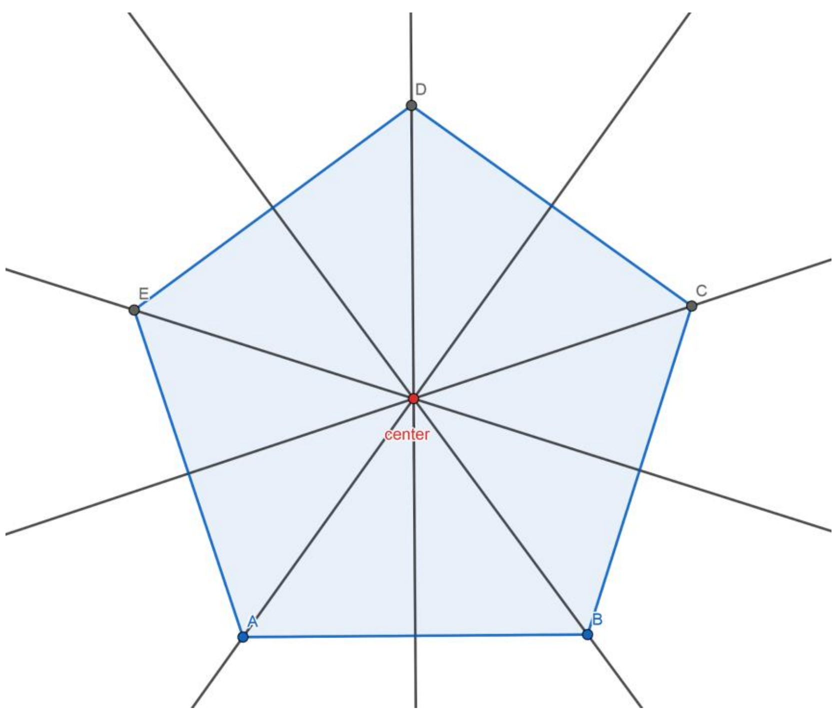


figure 3.2.2

3.3 Constructing The Image Polygon

To begin, construct the nine-point circle of each triangulated section in the original polygon. We have previously covered in chapter 2.2 how the mid-points of a triangle are actually concyclic and can be used to form the nine-point circle. Thus, by locating the mid-points of each side of the triangulated areas we can use these three points to produce the nine-point circle.

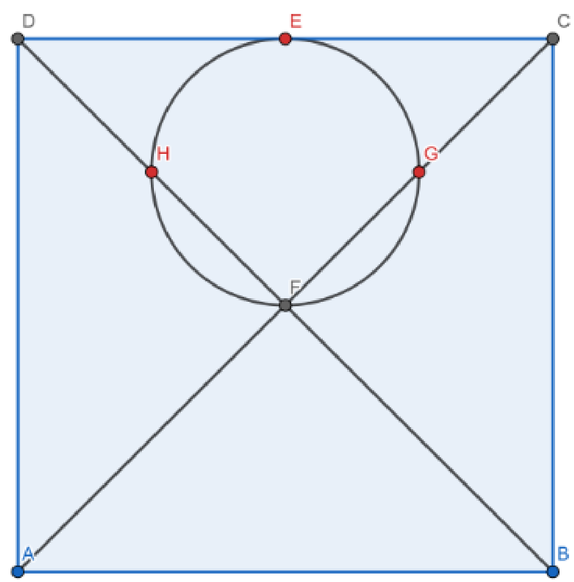


figure 3.3.1

Do this for each triangulated area and connect the centres of the nine-point circles to form the image polygon. From *figure 3.3.2* below, consider the area of square A'B'C'D' is 2 units^2 and the area of square ABCD is 16 units^2. Therefore square A'B'C'D' is homothetic with ABCD with a ratio of 1:8.

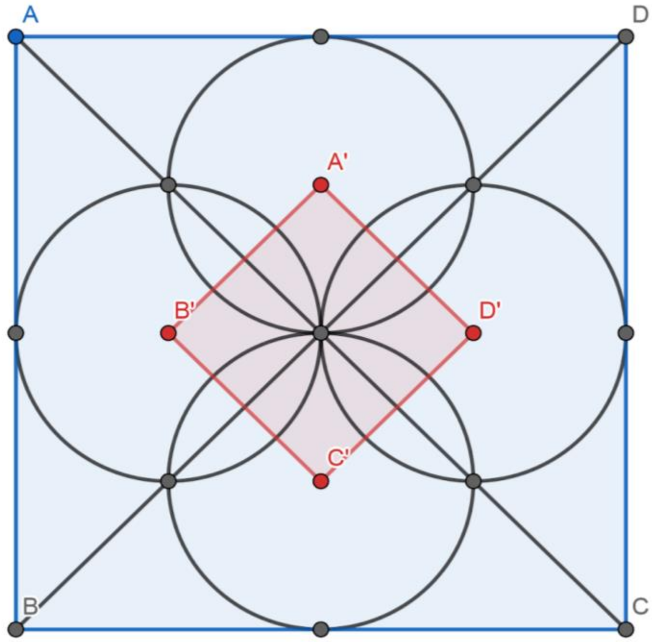


figure 3.3.2

3.4 Area Ratio Between the Original Polygon and Image Polygon

First, we constructed regular *n*-sided polygons, starting with a 4-sided polygon as shown in *figure 3.3.2* above. We can find the area ratio between the image polygon and the original polygon, and do the same for other *n*-sided polygons with n > 4. Here, $n = 3$ is not applicable as the nine point centres would all lie at the same point, this actively illustrates that the image polygon of a 3-sided polygon would have an area of *0* □□□□□^{2}as shown in *figure 3.4.1* below.

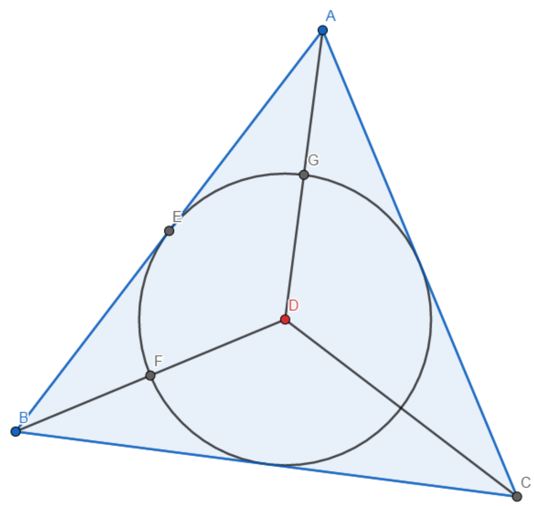


figure 3.4.1

The nine point circle of triangles *ABD*, *BCD* and *ACD* is circle *EFG* with centre at point *D*.

3.4.1 Regular Polygons

For regular *n*-sided polygons, we found that the area ratio between the original and image polygons decreases as *n* increases. To show this relationship, we have plotted a graph with the *x*-axis as *n* and the *y*-axis as the area ratio between the original and image polygons.

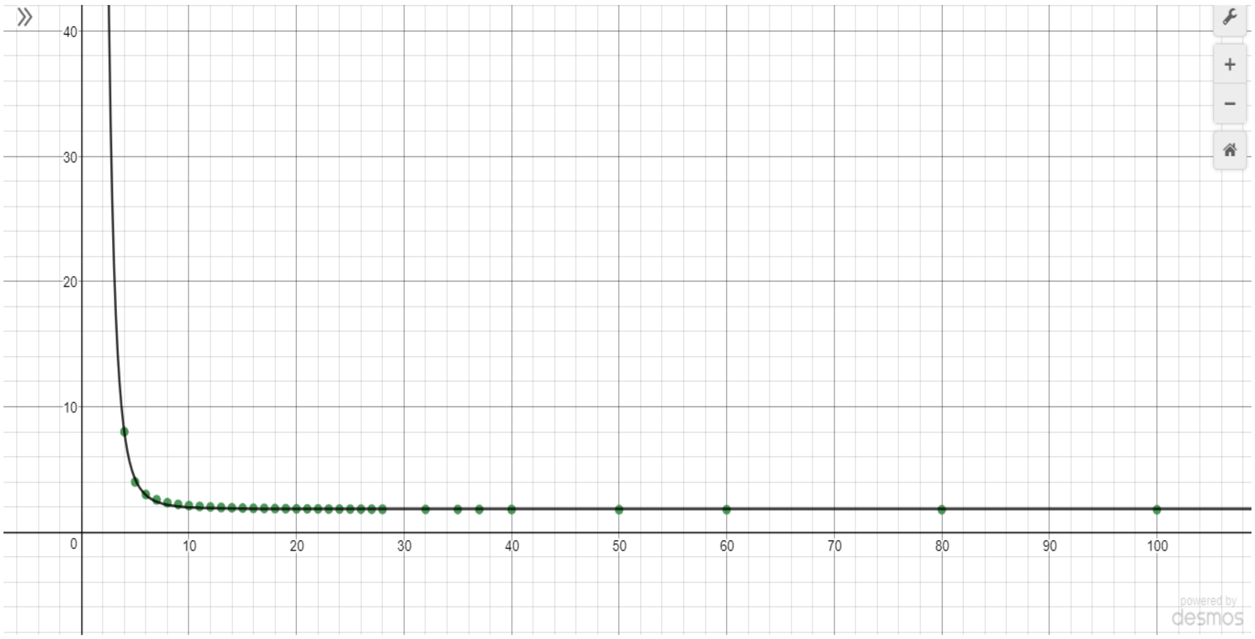


Figure 3.4.1.1

Based on the graph of *figure 3.4.1.1* above, we found the equation of the area ratio between the original polygon and the image polygon to be:

$$\square(\square) = 1.800332 + \frac{451.757800975}{7.332743 \times 10^{-5} + \square^{3.093611}}$$

3.4.2 Irregular Polygons

For irregular *n*-sided polygons, a fixed condition which we have set is that the included angles of the triangulated areas located between the lines connecting the centre to the vertices of the polygon are to be the same. To achieve this, we derive an irregular polygon from a regular polygon.

Consider a 4-sided regular polygon *ABCD*. Since this is an even-sided polygon, triangulate the polygon by constructing lines from one vertex to the opposite vertex, such as from A to C and from B to D. This time, extend these lines until it projects outside the regular polygon.

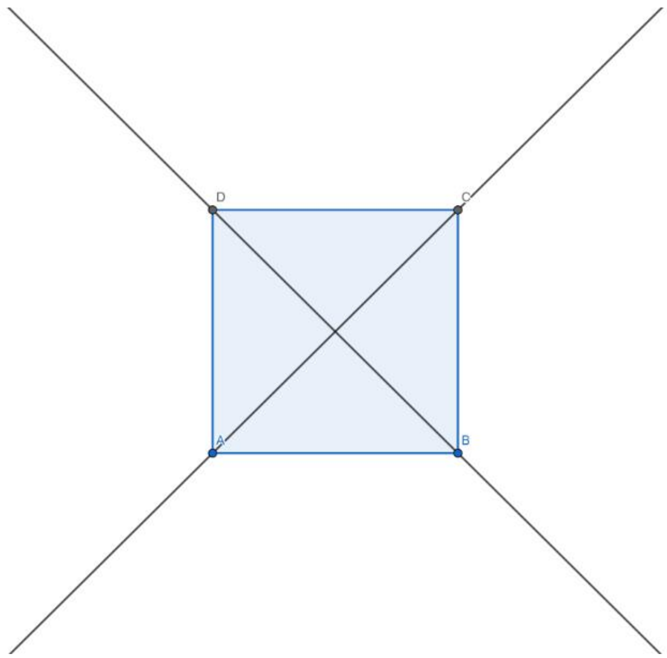


figure 3.4.2.1

Next, construct random points along the lines to make an irregular polygon out of the regular polygon. Instead of finding the centre of the irregular polygon EFGH, we will use the centre of the regular polygon ABCD, denoted by the red point *I* in *figure 3.4.2.2* below.

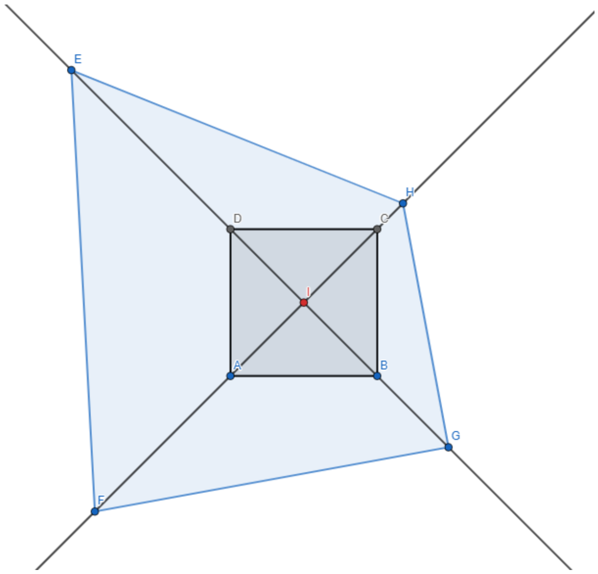


Figure 3.4.2.2

Next, construct nine-point circles inside the triangulated areas and connect the centres of the nine-point circles to form the image irregular polygon. Hence, we can find the area ratio.

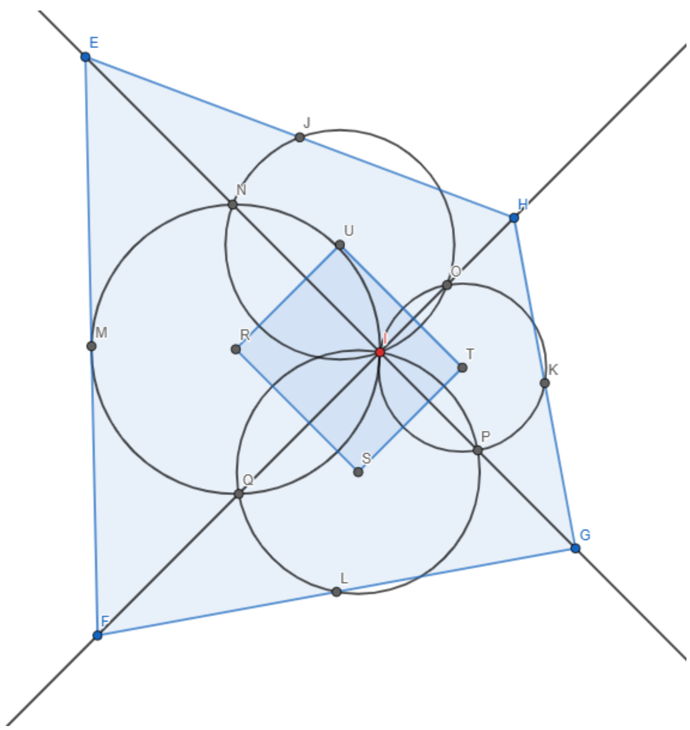
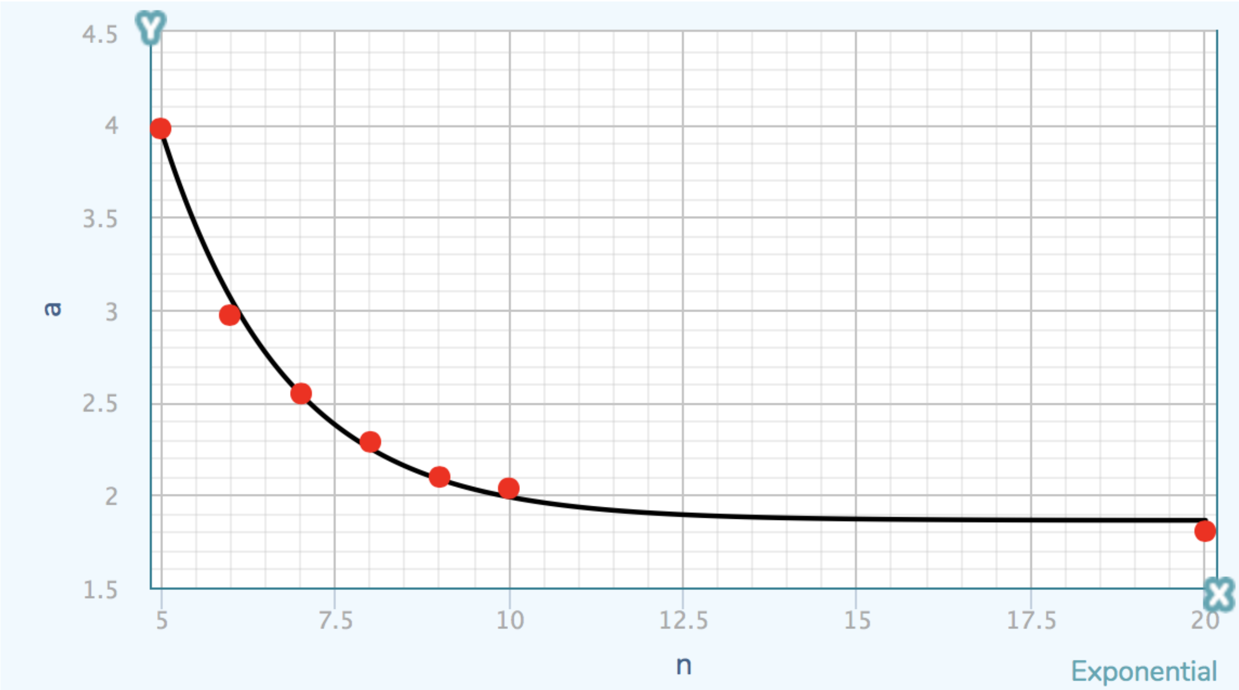


Figure 3.4.2.3

Just like with regular polygons, we found that the area ratio between the original and image polygons decreases as the *n* increases.



We found the equation of the area ratio between the original polygon and the image polygon to be:

$$\square(\square) = 1.760762 + \frac{407.470519184}{7.332743 \times 10^{-4} + \square^{3.256105}}$$

Chapter 4 - Results and Analysis

4.1 Analysis of Results

We have formed image polygons from the centers of *n* number of Nine-Point Circles in *n*-sided polygons. Now, we have found the relationship between the area of a regular polygon and the area of its image polygon, the relationship between the area of an irregular polygon and the area of its image polygon, and the difference between both polygon.

The area ratio between a regular polygon and its image polygon is:

$$\square(\square) = 1.800332 + \frac{451.757800975}{7.332743 \times 10^{-5} + \square^{3.093611}}$$

The area ratio between an irregular polygon and its image polygon is:

$$\square(\square) = 1.760762 + \frac{407.470519184}{7.332743 \times 10^{-4} + \square^{3.256105}}$$

Since there is very little difference between the two graph equations, we can consider the relationship of the area of a regular polygon with its image polygon and the area of an irregular polygon with its image polygon the same.

Chapter 5 - Conclusion

We studied the Nine-Point Circle which is a circle that passes through nine significant concyclic points defined from a triangle: the midpoint of each side of the triangle, the foot of each altitude, and the midpoint of the line segment from each vertex of the triangle to the orthocenter, where the three altitudes meet. Then we were curious as to how we could apply this Nine-Point Circle Theorem to *n*-sided polygons.

Inspired by the Homothetic Transformation and a diagram of a Triangle formed by the Centers of Three Nine-Point Circles, we divided polygons (which we called the "original polygon") into *n*-triangles by constructing lines from the centre of the polygon to its vertices, and connected the centres of the nine-point circles of each triangle to form another polygon (which we called the "image polygon"). A formula connecting the area of a regular *n*-sided polygon to the area of its image polygon was found: $\square(\square) = 1.800332 + \frac{451.757800975}{7.332743\times 10^{-5} + \square^{3.093611}}$. Another formula was derived from the area ratio between an irregular *n*-sided polygon and its image polygon: $\square(\square) = 1.760762 + \frac{407.470519184}{7.332743\times 10^{-4} + \square^{3.256105}}$, and that the relationship of both equations can be considered the same due to its small difference.

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