The Kakeya Needle Problem

Paya Lebar Methodist Girls School (PLMGSS)

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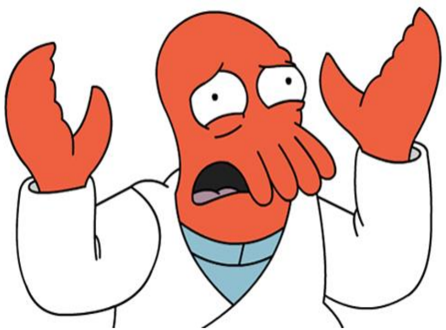
Why Kakeya Needle problem?



Objectives

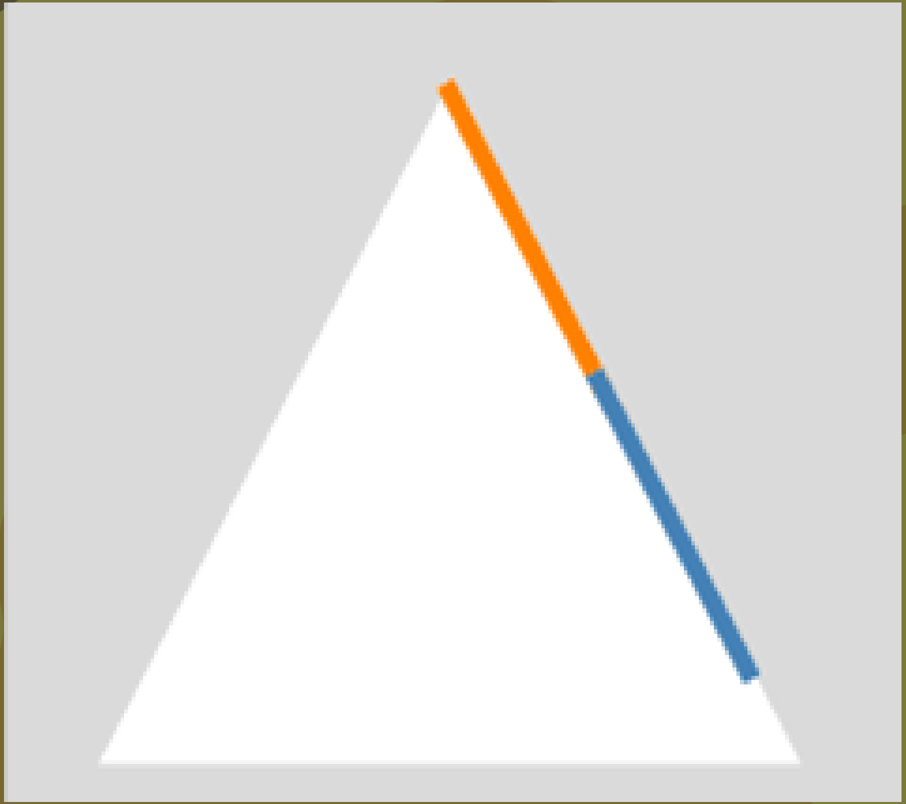
To derive a general formula to find the area of the Besicovitch solution.

To verify does the Besicovitch's solution have a smaller region for the needle to turn in 360° compared to the Kakeya's solution .

Understanding how the needle will turn 360° in following areas

①

Equilateral triangle



By Pythagoras's theorem,

$$1^2 + \left(\frac{1}{2}k^2\right) = k^2$$

$$k = \frac{2}{\sqrt{3}}$$

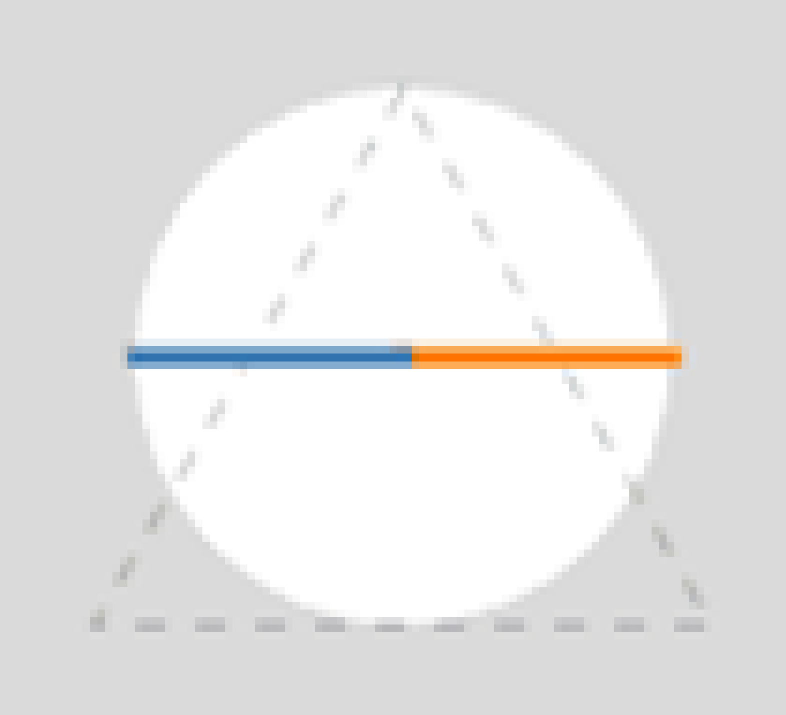
$$\text{Area of region} = \frac{\sqrt{3}}{4}k^2$$

$$= \frac{\sqrt{3}}{4} \times \frac{2^2}{\sqrt{3}^2}$$

$$= \frac{1}{\sqrt{3}}$$

②

Circle



$$\text{Area of region} = \pi\left(\frac{1}{2}\right)^2$$

$$= \frac{\pi}{4}$$

③ Reuleaux triangle

$$Area_{segment} = \frac{1}{2}r^2(\theta - sin\theta)$$

$$= \frac{1}{2}r^2(\frac{\pi}{3} - sin\frac{\pi}{3})$$

$$= \left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right)r^2$$

$$Area_{equilateral\ traingle} = \frac{\sqrt{3}}{4}r^2$$

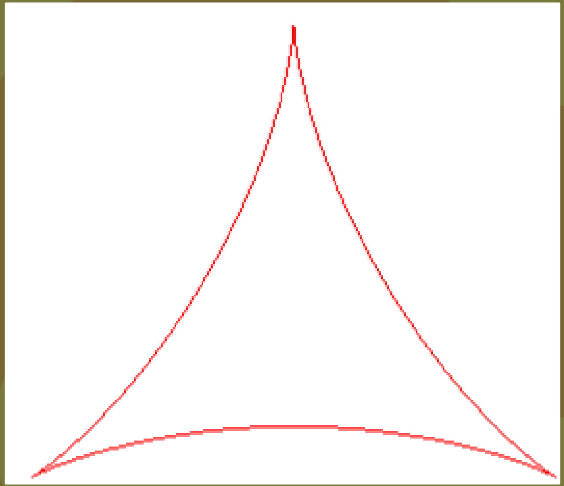
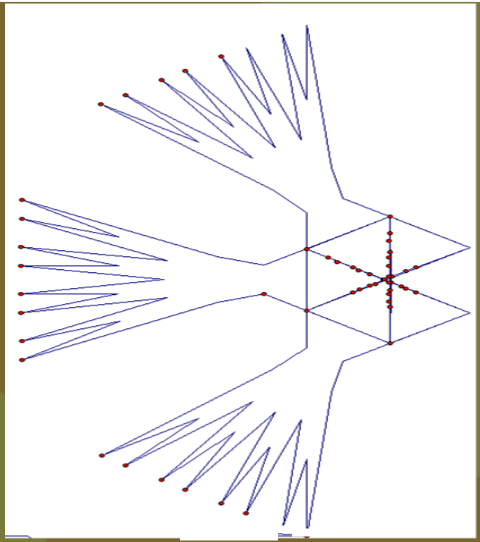
$$Area\ of\ region = 3Area_{segment} + Area_{equilateral\ traingle}$$

$$= 3\left(\frac{2\pi - 3\sqrt{3}}{12}\right)r^2 + \frac{\sqrt{3}}{4}r^2$$

$$= \frac{(\pi - \sqrt{3})r^2}{2}$$

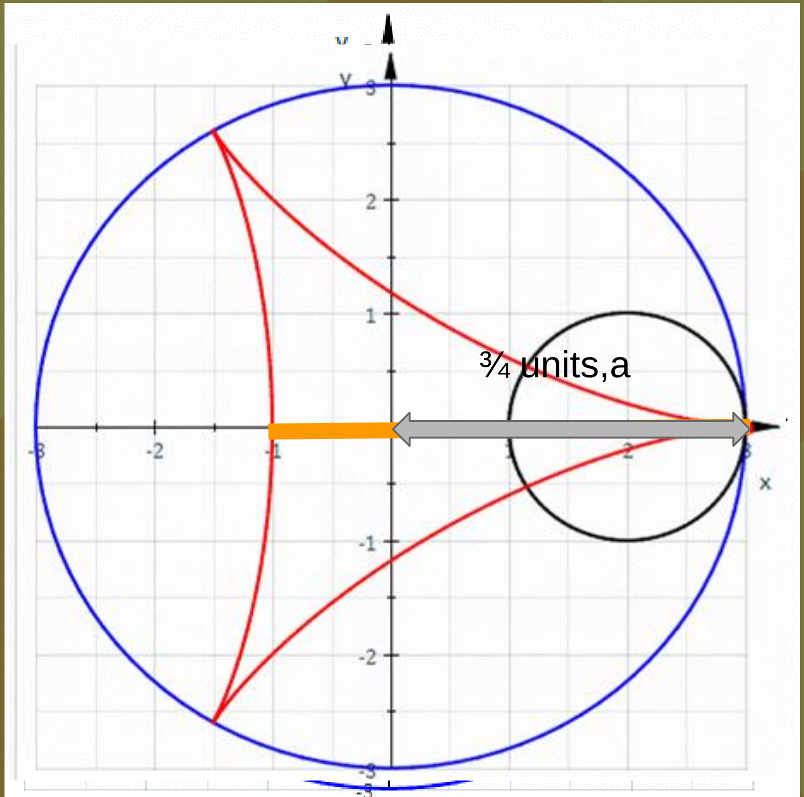
$$= \frac{(\pi - \sqrt{3})}{2}$$

Proposed solutions

Mr Kakeya:	*Mr Besicovitch:*
	
Deltoid	Perron Tree

④

Deltoid



Since delt … s

Given that a … l area

formula of

$area_n = \frac{(n-}{}$

For a deltoic

Substituting $a = \frac{3}{4}$,

$$a^2 = \frac{9}{16}$$

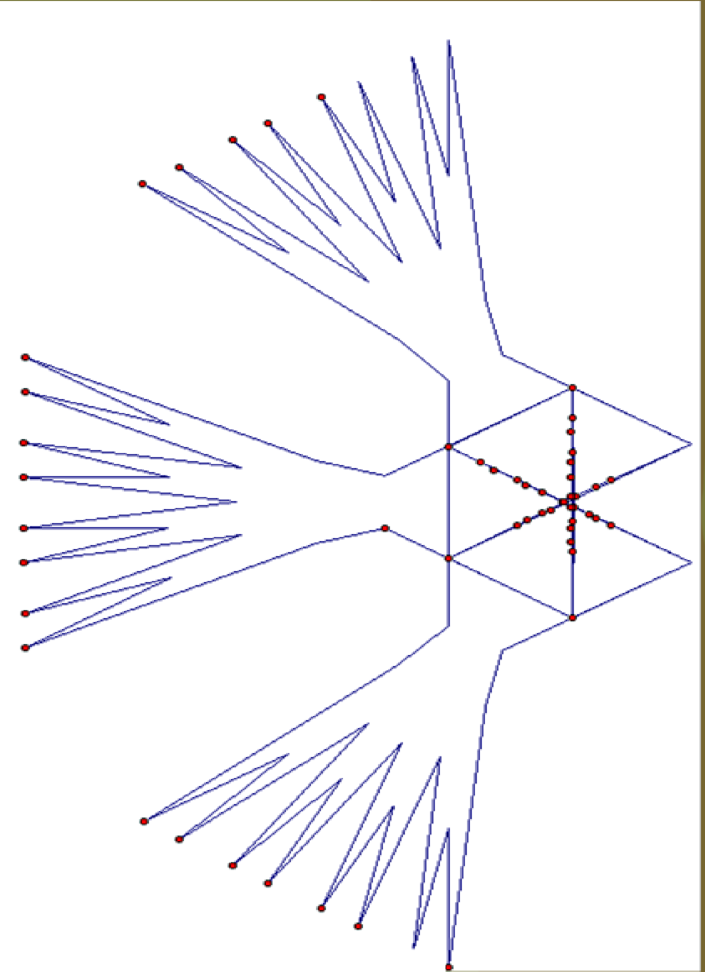
$$= \frac{1}{8} \times \frac{9}{2}$$

Therefore, area of deltoid is

$$\frac{2}{9}\pi a^2 = \frac{\pi}{8}$$

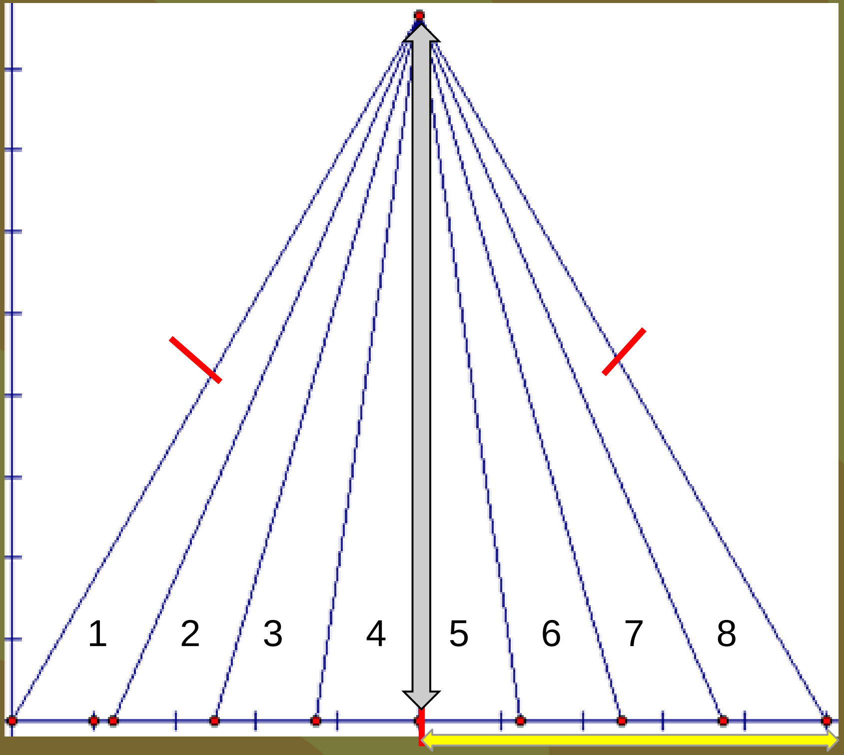
5

Perron Tree

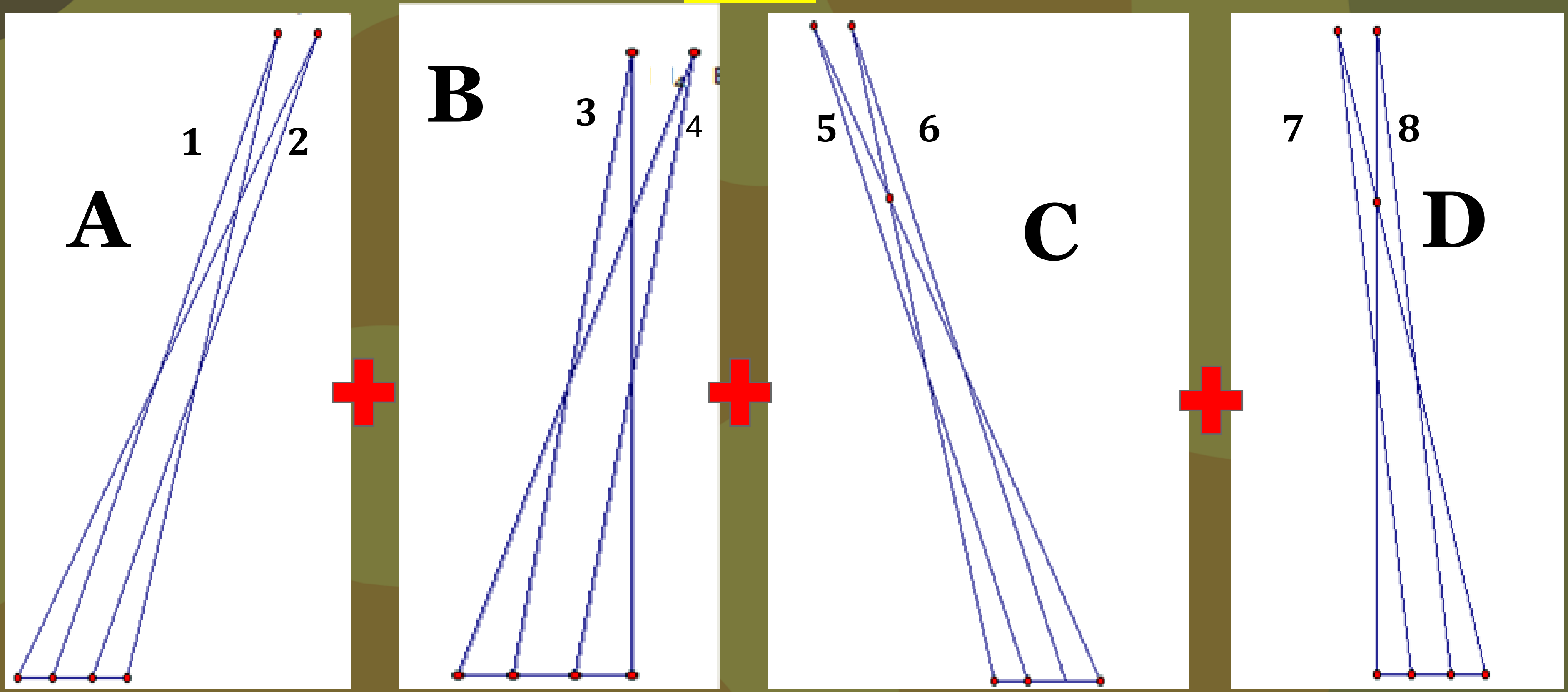


Besicovitch set construction and movement of needle

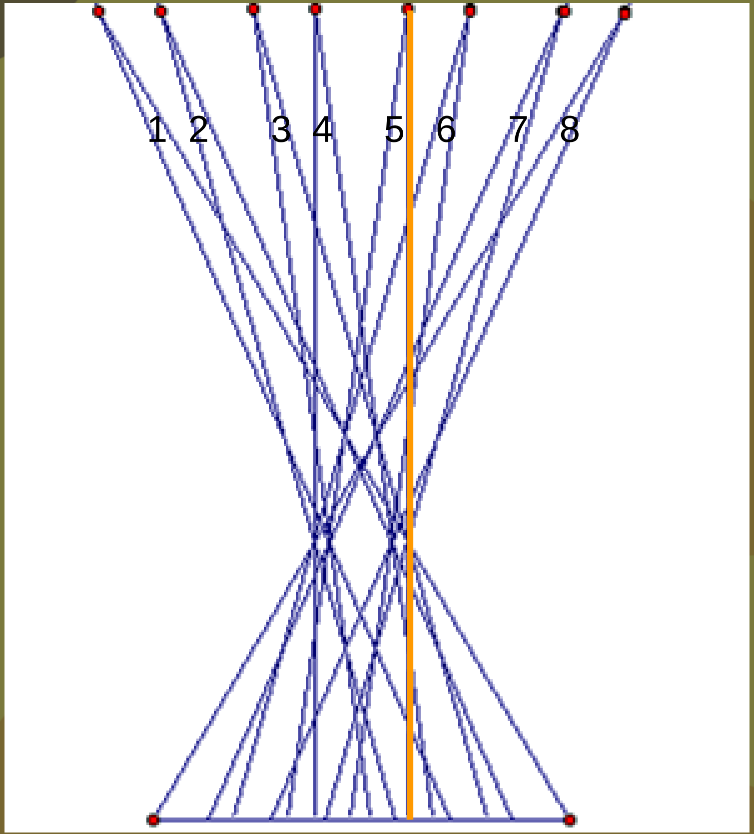
Construction of the Besicovitch set



Construction of the Besicovitch set



Construction of one Besicovitch Set



A Consists of 1 and 2

C Consists of 5 and 6

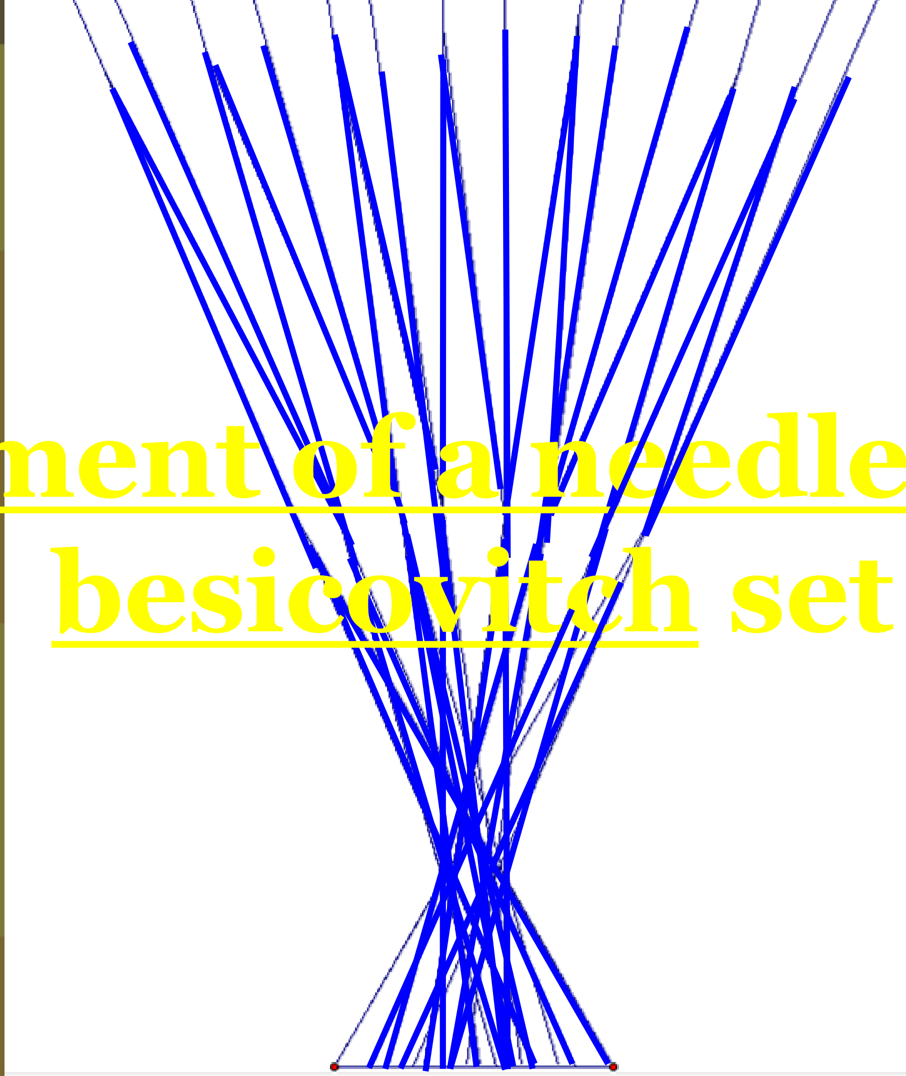
B Consists of 3 and 4

D Consists of 7 and 8

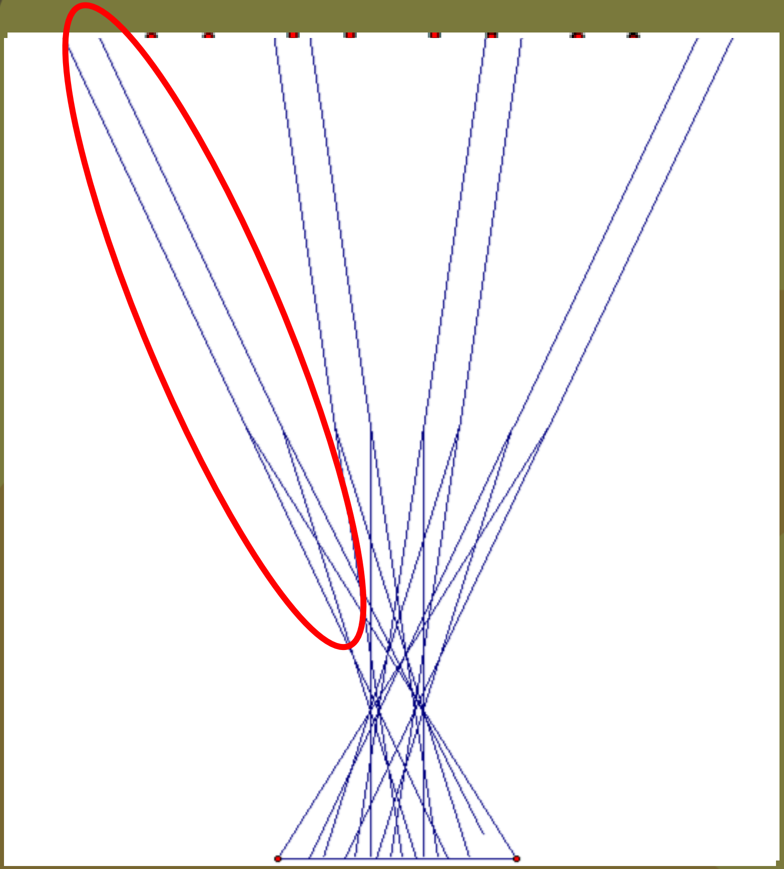
Legend:

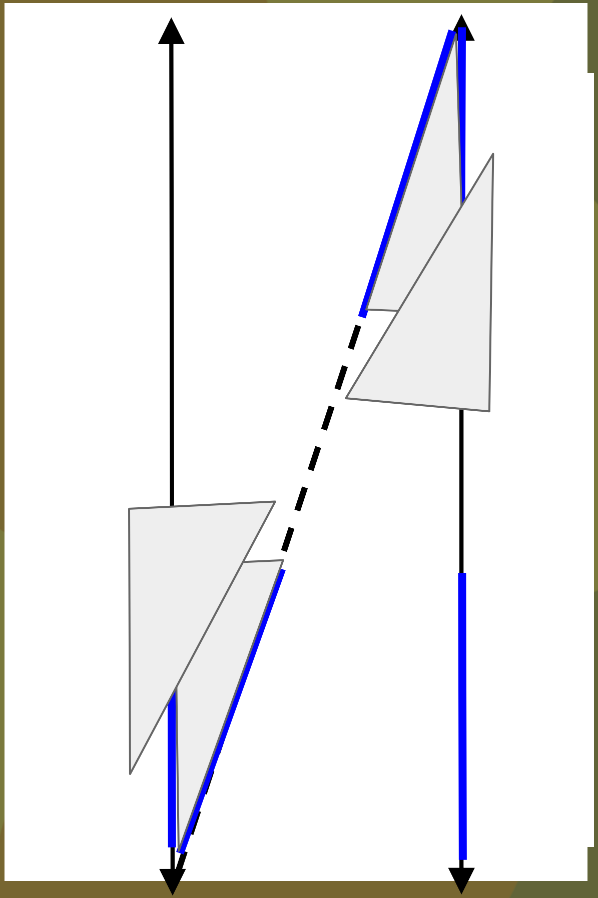
Length of needle
=
Height of piece 4 or 5

Movement of a needle in one besicovitch set

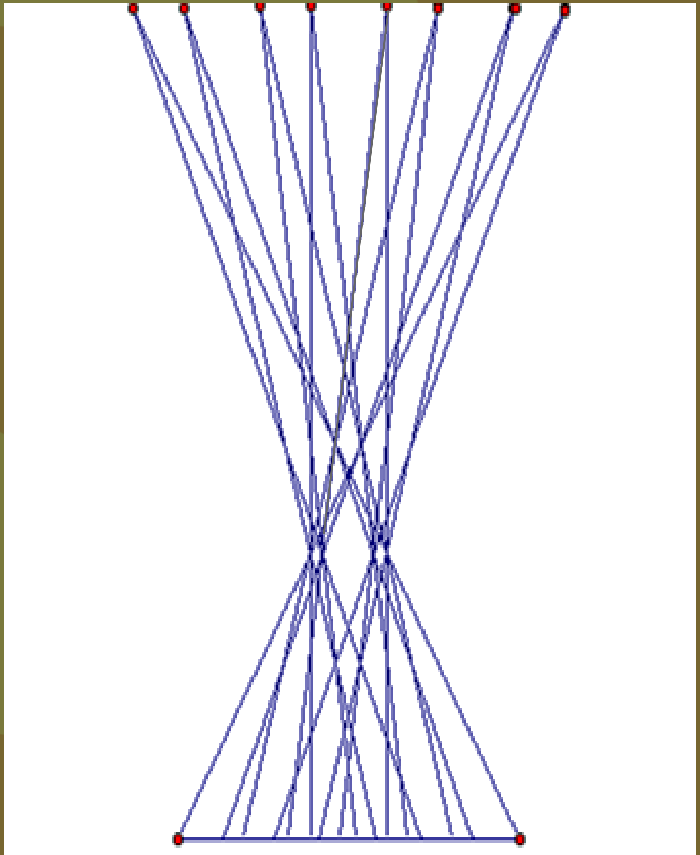


Movement of needle

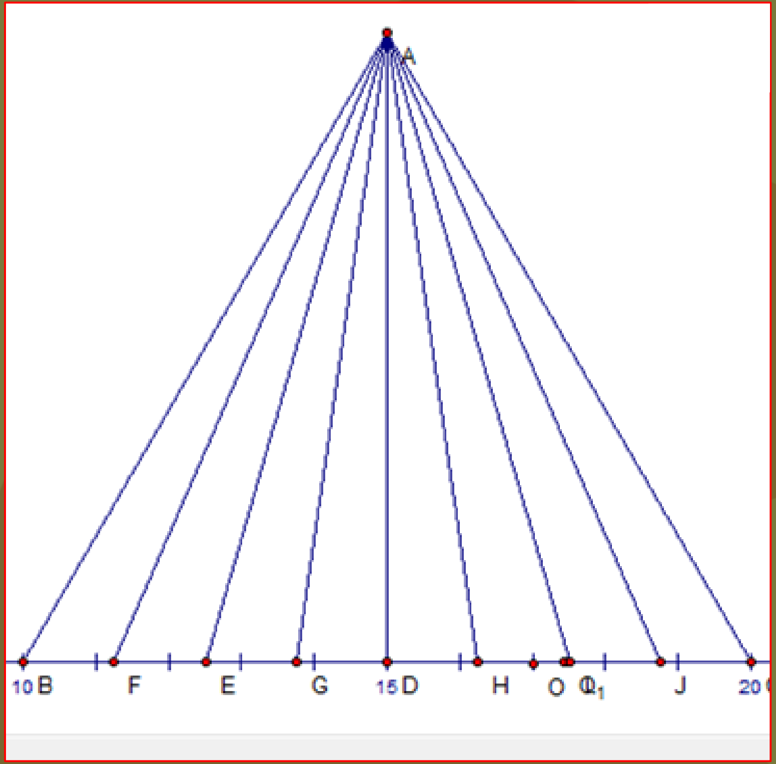




One Besicovitch Set



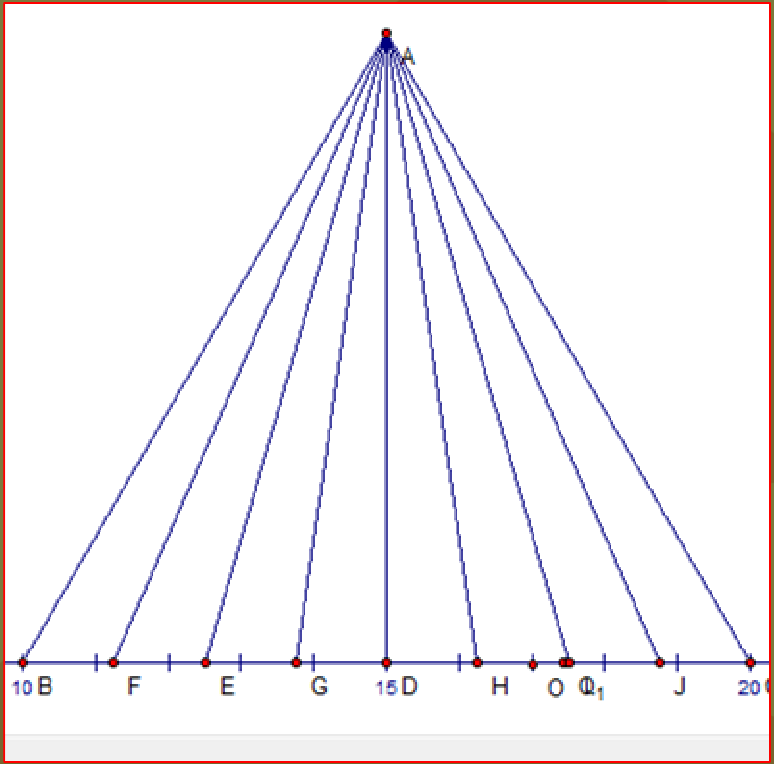
Area of Besicovitch Set

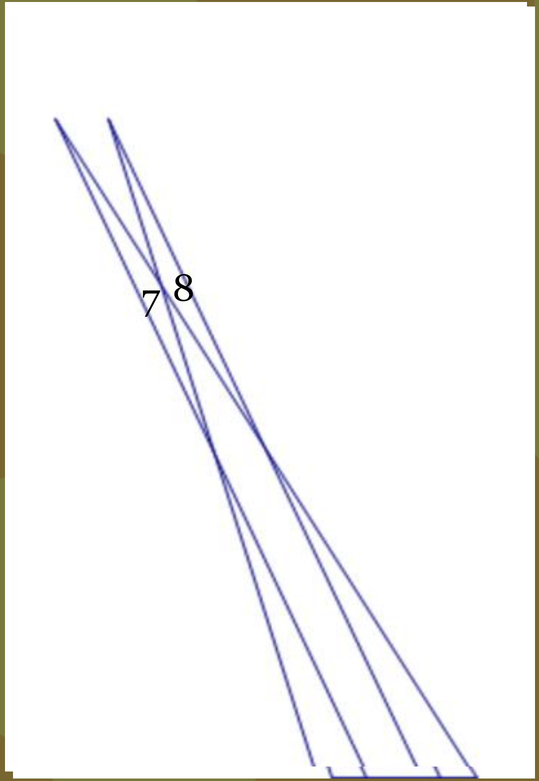


Let x=10

Area of equilateral triangle
= ½ x base x perpendicular height
=½ x 10 x 8.66025
= 43.301 (5 s.f.)

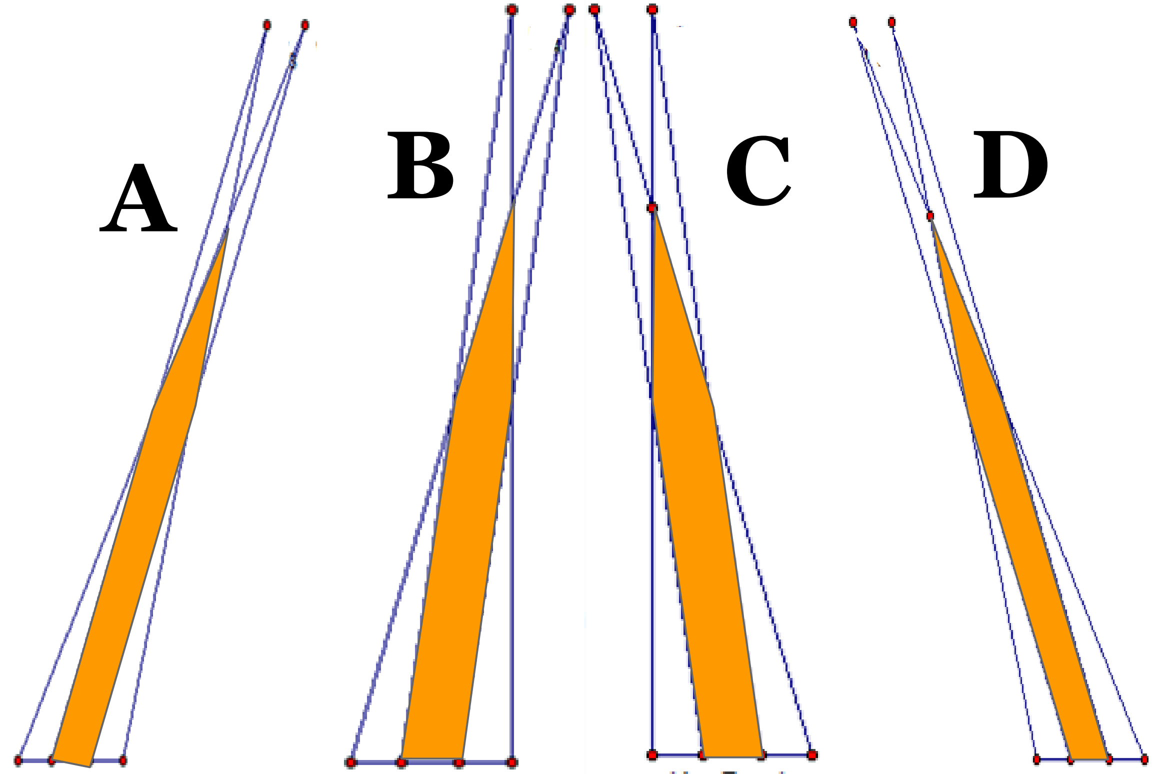
Forming of the biggest overlap for 2 pieces





Base length=1.25

Overlapping area of two pieces of triangles



Total overlapping area of pairs of triangles

$= Area_A + Area_B + Area_C + Area_D$

$= 3.608 + 3.635 + 3.585 + 3.623$

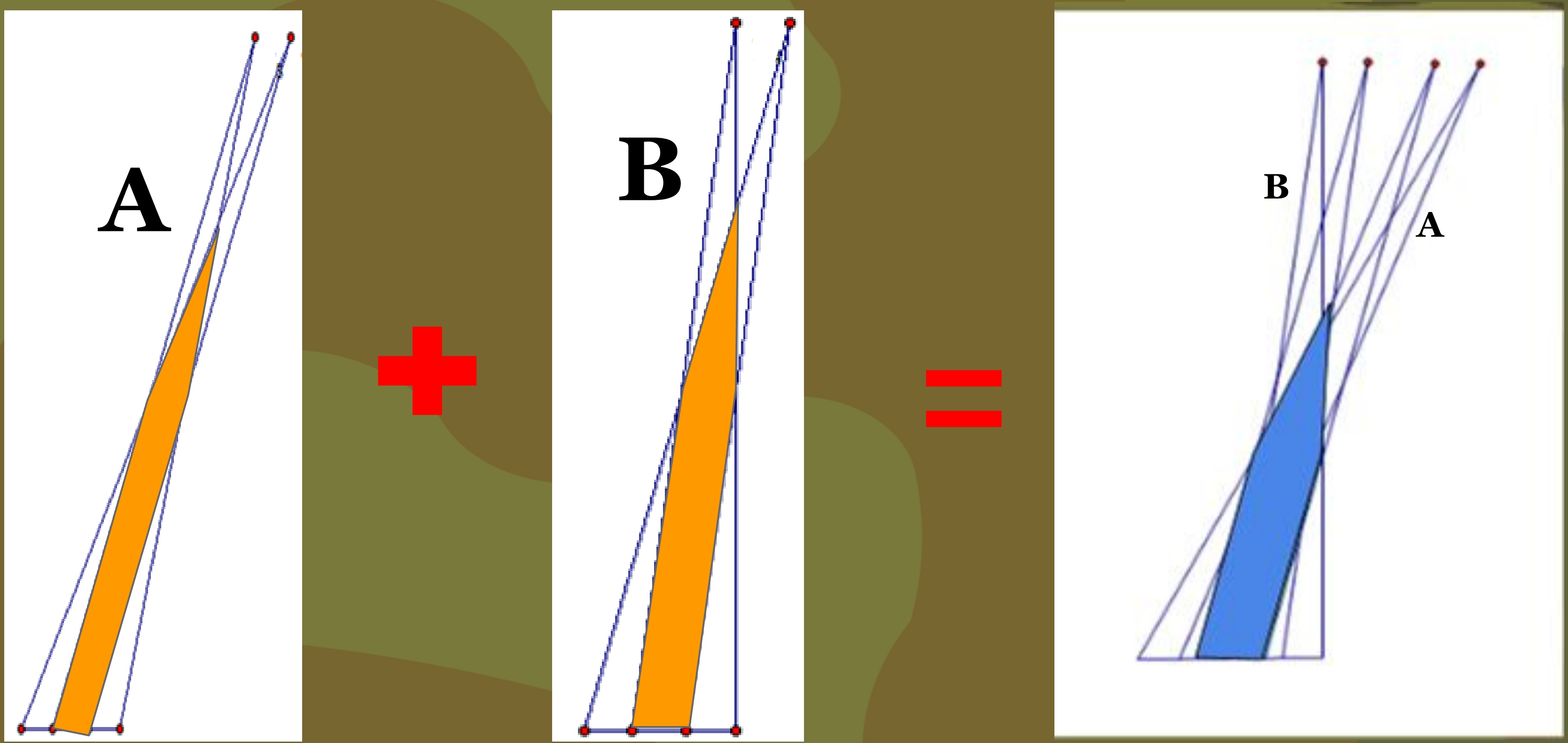
$= 14.450 \text{ cm}^2$

Total leftover area of pairs of triangles

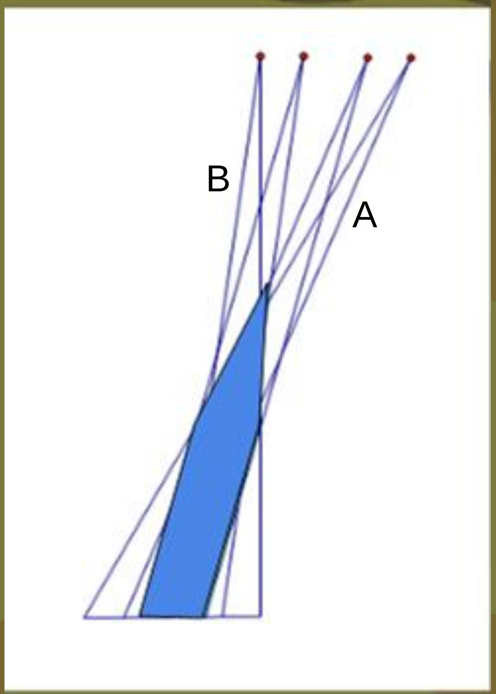
$= 43.301 - 14.450$

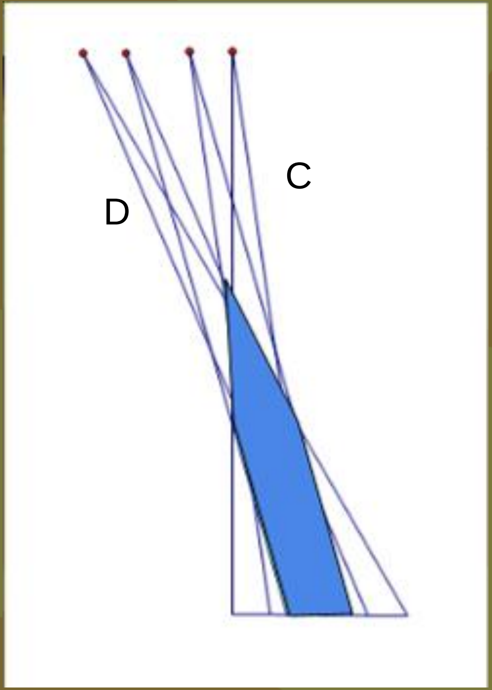
$= 28.800 \text{ cm}^2$

Forming of largest overlap for 2 pairs of triangles



Overlapping and leftover areas of two pairings of triangles





Total overlapping area for 2 pairings

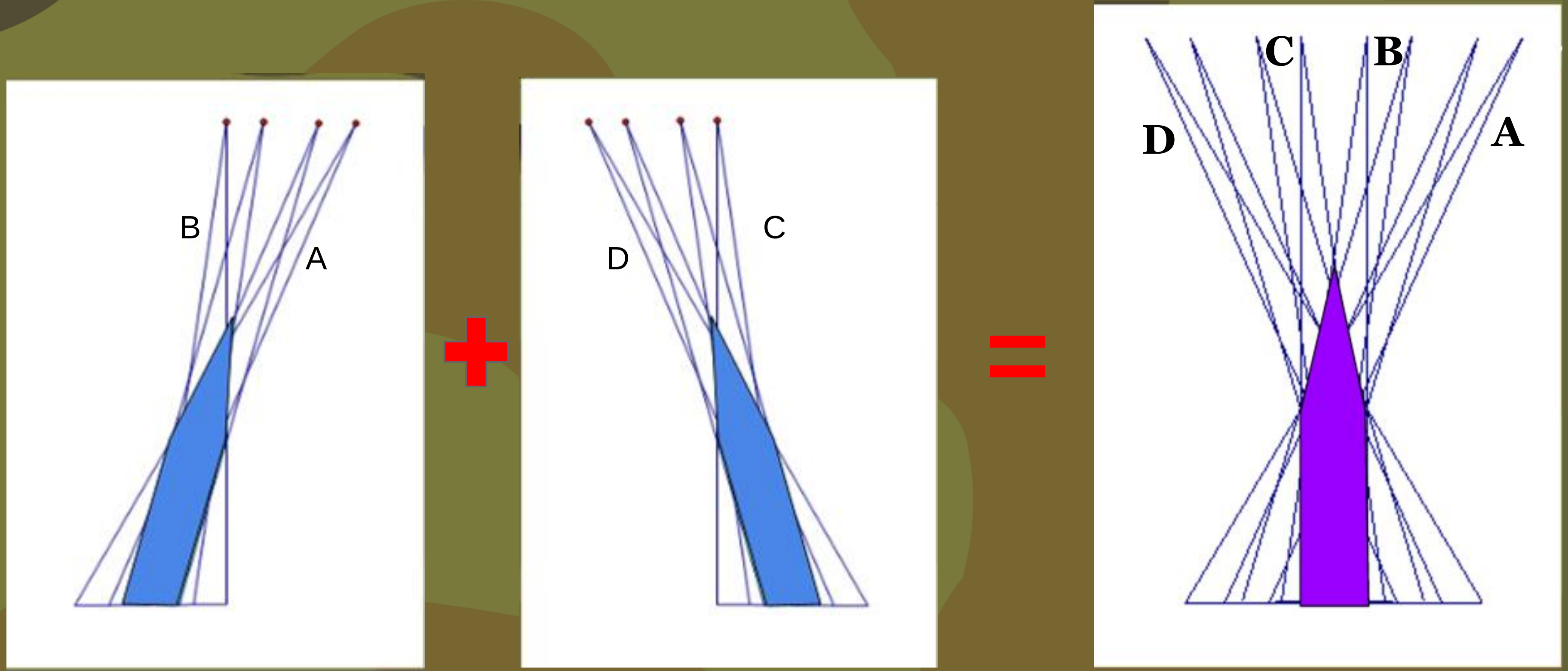
= 14.408

Total leftover area for 2 pairings

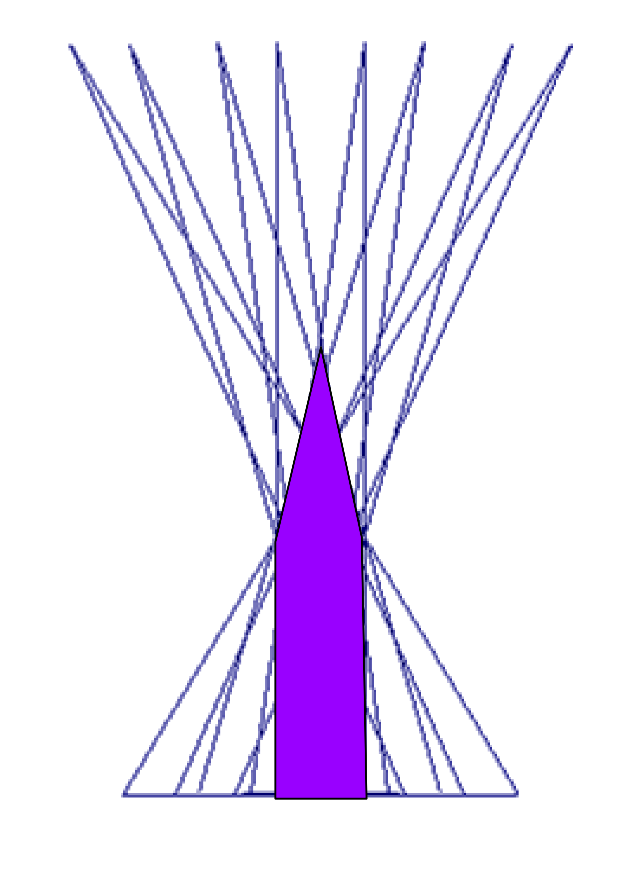
= 28.800 – 14.408

= 14.391

Forming of Besicovitch set



Overlapping and leftover areas for Besicovitch set



Overlapping area for 1 besicovitch set

$$= (\frac{1}{2} \times 2.32 \times 0.99) + (2.96 \times 0.99)$$

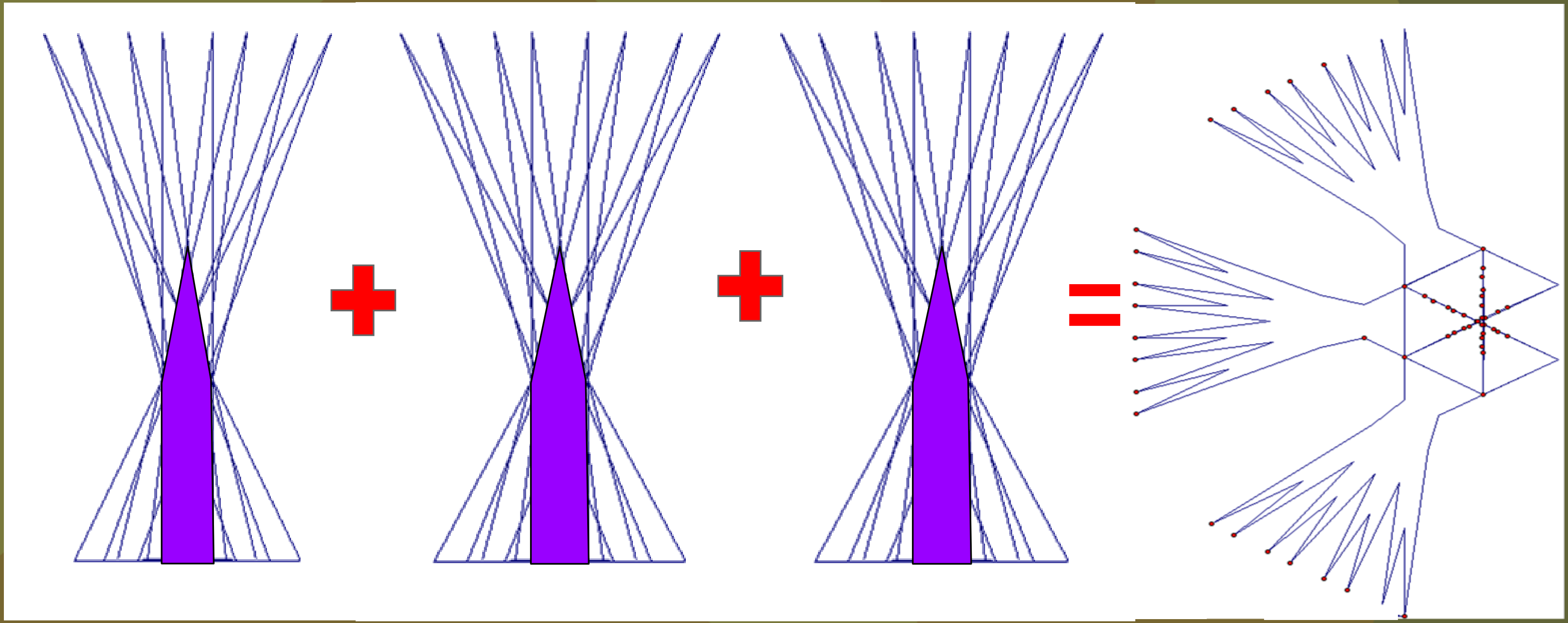
$$= 4.0788$$

Total Leftover area for 1 besicovitch set

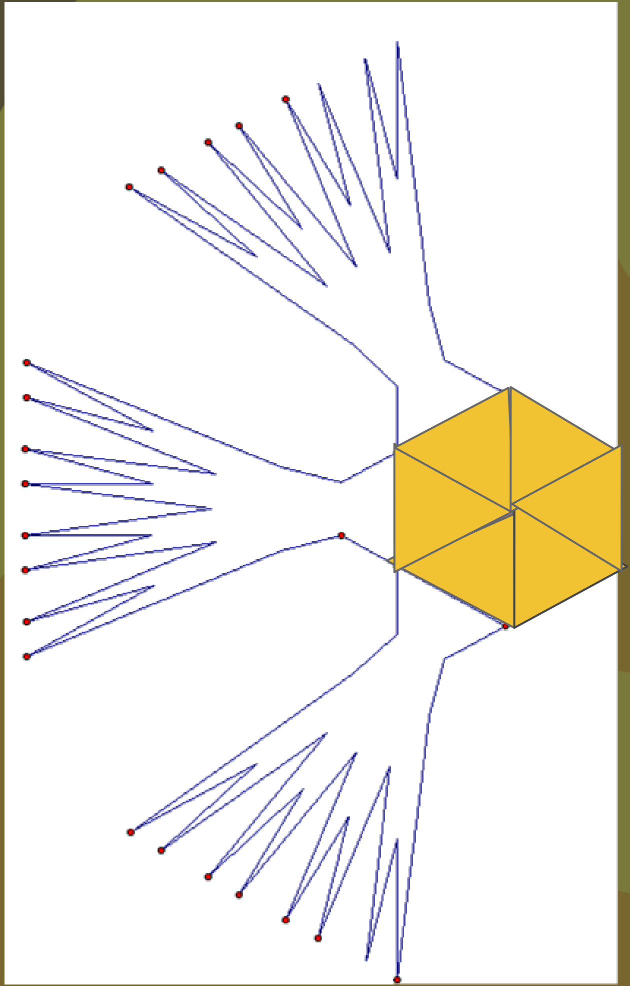
$$= 14.391 - 4.0788$$

$$= 10.313$$

Forming of the Perron Tree



Overlapping and leftover areas for Perron Tree



Total overlapping area for 3 Besicovitch sets

$$= [[\frac{3\sqrt{3}}{2}(2.198)^3] \div 6] \times 4$$

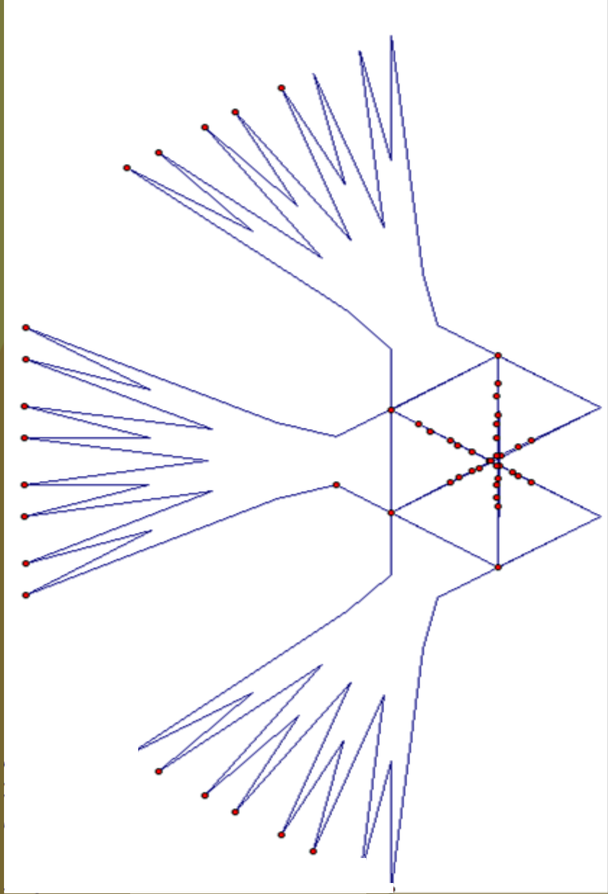
$$= 8.36789$$

Total leftover area for 3 Besicovitch sets

$$= 10.313 \times 3$$

$$= 30.93866$$

Area of Perron Tree



Total overlapping area for Perron Tree

$$= [[\frac{3\sqrt{3}}{2}(2.198)^{3}] \div 6] \times 4$$

$$= 8.36789$$

Total area for Perron Tree

$$= 10.313 \times 3$$

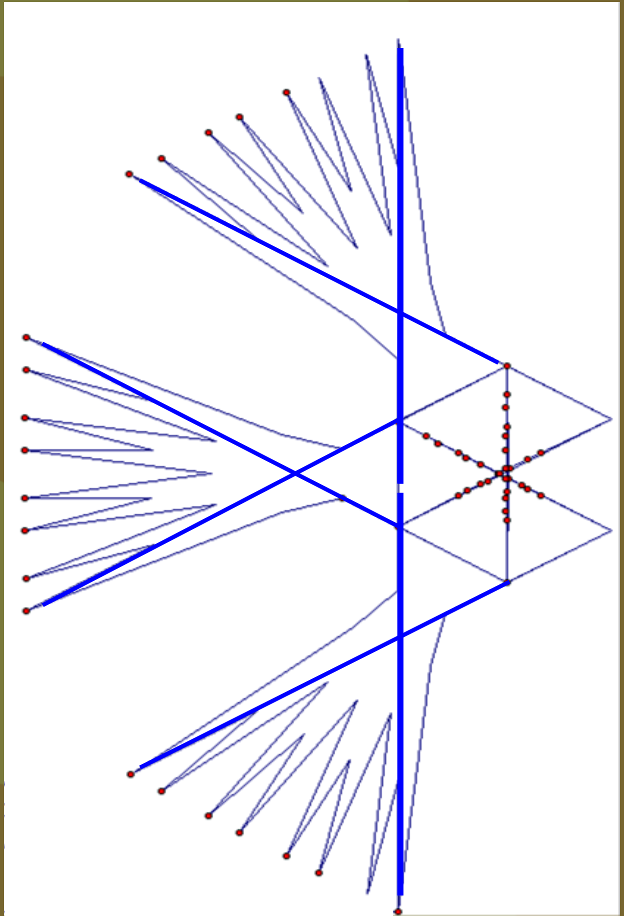
$$= 30.93866$$

Total Leftover for Perron Tree

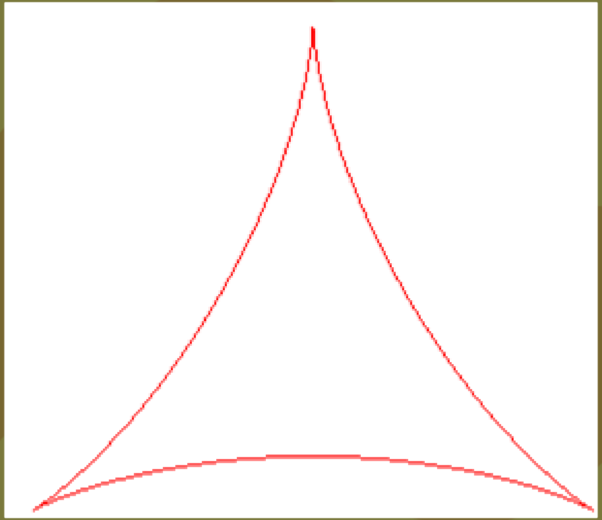
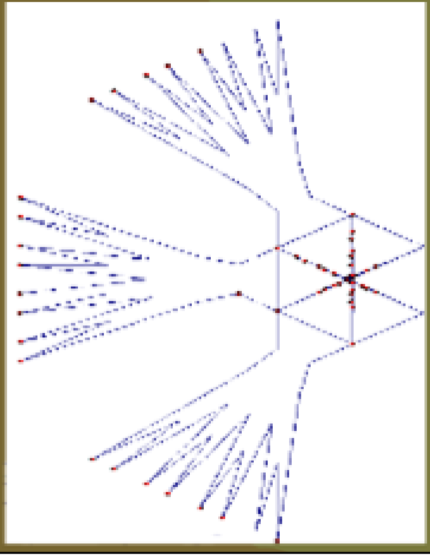
$$= 30.93866 - 8.36789$$

$$= 22.57077$$

Movement of needle in a perron tree



COMPARISON OF DELTOID AND PERRON TREE

DELTOID	PERRON TREE
	
$Area = \frac{2}{9}\pi\left(8.66 \times \frac{3}{4}\right)^2$ $= 29.5\ cm^2$	$Area = 22.5$

General Formula of Perron Tree

Since $k = $ *side of equilateral triangle,* $S = \frac{3k}{2}$,

Excluding the overlapping area,

$\therefore$ *Total area of Perron Tree*

$$= \frac{521}{1000} \times \sqrt{s(s-k)^3}$$

$$= \frac{521}{1000} \times \sqrt{\frac{3k}{2}\left(\frac{3k}{2} - k\right)^3}$$

$$= \frac{521}{1000} \times \sqrt{\frac{3k}{2}\left(\frac{k}{2}\right)^3}$$

$$= \frac{521}{1000} \times \sqrt{\frac{3k^4}{16}}$$

$$= \frac{521}{1000} \times \frac{\sqrt{3}k^2}{4}$$

$$= \frac{521\sqrt{3}k^2}{4000}$$

$$Area\ of\ Perron\ Tree = \frac{521\sqrt{3}k^2}{4000}$$

Conclusion

Area of Perron Tree < Area of deltoid

Moreover, we found that

$$Area\ of\ Perron\ Tree = \frac{521\sqrt{3}k^2}{4000}$$

Limitations

- We are not able to determine the exact value of the area, as the overlapping area is generalised.

- Since the Perron tree is only enable to turn the needle 360 degrees using the parallel lines, it may not be feasible in the real world context of moving soldiers through the smallest area.

Further Extension on Research

Subdivide the perron tree into 2^n triangles, where n=1, 2, 3, 4,5 etc. (E.g. 8 triangles, 16 triangles, 32 triangles) to construct more of such trees then connect to make up a sequence of trees which can converge to something, in the limit, which has interesting properties where you can test for area zero, 0.

THANK YOU

Q & A

1. For needle to rotate 360^ it must move out of set, is that practical for real life soldiers?

It depends on the space they have to rotate in, however, Mr B and many online sources have claimed that this is how the needle will have to rotate and he did think of this problem while thinking of the Japanese soldiers' movements back then.

1. Scale might be bigger, so what happens to the area of the B.set?

Ratio of map scale to actual is proportionate.

1. Why must the needle move out of the set?

MrB and online sources have stated so, the needle moving along the parallel lines take up less area for movement.

1. Parallel line extension:

To be able to move from one triangle to the other to complete a round of 180 degrees, the needle has to move out of the set to be . able to make that 'jump'. Since no area will be covered when the needle travels in a parallel line segment and causes the area of the 'jump' to be smaller when compared to just moving across to the other parallel line, the needle has to move out of the set up but it is considered to be still within the restricted parts of the set up

Q & A

5. Question: why the approach in ur report cannot be also used in ur current area calculation?

In our previous report, we assume each layer has the same height, thus able to find the area of triangles, which is not true is our current construction. Each layer of triangles is not having the same height, thus not able to calculate directly using the usual formula of the area of triangle. We calculate area of different triangles of the overlapping area using Herons formula, and subtract the overlapping area from the original equilateral triangle, through using of the GSP. The new approach leads to more accurate value of the area of the besicovitch set.