INTERGALACTIC GRAVITY

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ABSTRACT

In this project, our group has explored the ways that Mathematics can be applied in Astronomy to do calculations using formulas and various theorems. The topic that we have focused on is gravity, how important it is in our solar system and its impacts, and we would also be covering topics on Newton's law of gravitation, Einstein's general theory of relativity, black holes, and the no hair theorem and Kerr metric used in relation with black holes.

We hope to provide people with a greater understanding on the beauty and complexity of our universe and how Mathematics has been used to help scientists to learn more about the objects in our space.

1. Introduction

Astronomy is the scientific study of celestial objects such as stars or planets, and phenomena that originate outside the Earth's atmosphere. It is concerned with the evolution, physics, chemistry, meteorology, and motion of celestial objects, as well as the formation and the development of the universe. This is an extract of the definition of 'astronomy' taken from Wikipedia. How then is mathematics used in astronomy? Mathematics can be used by astronomers in many different ways. One example is when astronomers are forming and testing theories of physical laws that govern objects in the sky. These equations are manipulated using mathematics. Of course, there are many other ways in which mathematics can be applied in astronomy, and we would soon be looking at some of them.

1.1 Gravity

Gravity, commonly known as gravitation, is a natural phenomenon by which objects with mass attract each other. Every planetary body is surrounded by its own gravitational field, and assuming there is a spherically symmetrical planet, the strength of its field would be proportional to its body mass and inversely proportional to the square of the distance from the centre of its body. The strength of the gravitational field is numerically equivalent to the acceleration of objects under its influence. For example, the value of the gravity at the Earth's surface is approximately $9.8m/s^2$. This basically means that if we ignore air resistance, with every second, an object falling near the Earth's surface would increase its velocity by 9.8m/s. Also, if air resistance is ignored, all objects, if released from the same height, would fall and hit the ground at the same time.

Gravity also determines how the planets revolve around the sun. The Earth, for example, following Newton's First Law, would have actually traveled in a straight line. However, due to gravity, the force of attraction between the sun and the earth is actually large enough to force the earth to veer off from its original intended track, and to make it follow an elliptical orbit around the sun. As the gravitational pull of an object is largely based on its mass, the sun, compared with the other planets, would exert the largest gravitational pull. Thus, the sun would pull the planets towards it. However, the reason why the planets still stay orbiting in space is because they have enough sideways motion of their own to stay 'mid-space' and not plunge down into the sun. However, though gravitation and gravity are mostly interchangeable in everyday use, a distinction in made in scientific usage. "Gravitation" is a general term which describes the phenomenon where bodies with mass are attracted to one another. "Gravity", on the other hand, refers specifically to the net force exerted by Earth on objects in its vicinity as well as by other factors, such as the Earth's rotation.

2. Newton's Laws Of Gravitation

There are three physical laws of motion that form the basis for classic mechanics. The first is a body at rest stays at rest, and a body in motion stays in motion, unless it is acted on by an external force. Next, force is proportional to mass times acceleration. Equivalently, force is proportional to the time rate of change of momentum. Lastly, whenever a first body exerts a force (F) on a second body, the second body exerts a force (-F) on the first body. Both forces are equal in size and opposite in direction.

2.1 Newton's First Law

There is a set of inertial reference frame relative where all particles with no net force acting on them will move without change their velocity or a body persists its state of rest or of uniform motion unless acted upon by an external unbalanced force. This law is often referred to as the law of inertia. Due to this law, an object at rest will remain at rest while an object in motion will remain in motion unless acted on by an unbalanced force.

2.2 Newton's Second Law

The net force on a particle is equal to the time rate of change of its linear momentum:

$$\mathbf{F} = \frac{\mathrm{d}(m\mathbf{v})}{\mathrm{d}t} = m\,\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}$$

where **F** is the force vector, *m* is the mass of the body, **v** is the velocity vector and *t* is time. The product of the mass and velocity is momentum or quantity of motion. This equation shows the physical relationship between force and momentum for a body with constant mass.

2.3 Newton's Third Law

When a particle exerts a force on another particle, the second particle simultaneously exerts a force with the same magnitude in the opposite direction. Simplified, it is said that to every action there is an equal and opposite reaction. This law means that all forces are interactions and so there is no such thing as a unidirectional force.

2.4 Newton's Law Of Universal Gravitation

It is an empirical physical law that describes the gravitational attraction between bodies with mass. Every point mass attracts every other point mass by a force pointing along the line intersecting both points. The force is proportional to the product of the two masses and inversely proportional to the square of the distance between the point masses:

$$F = G\frac{m_1 m_2}{r^2},$$

where F is the magnitude of the gravitational force between the two point masses, G is the gravitational constant (approximately 6.673×10^{-11} N m^2 kg^{-2}), m_1 is the mass of the first point mass, m_2 is the mass of the second point mass and r is the distance between the two point masses.

2.4.1 Gravitational Field

The gravitational field is a vector field that describes the gravitational force which would be applied on an object in any given point in space, per unit mass. It is equal to the gravitational acceleration at that point. This can be written as a vector equation to account for the direction of the gravitational force as well as its magnitude. Here, quantities in bold represent vectors.

$$\mathbf{F}_{12} = -G\frac{m_1 m_2}{|\mathbf{r}_{12}|^2}\,\hat{\mathbf{r}}_{12}$$

Where $\mathbf{F}_{12}$ is the force applied on object 2 due to object 1. G is the gravitational constant; m_1 and m_2 are the masses of objects 1 and 2 respectively, $|\mathbf{r}_{12}| = |\mathbf{r}_2 - \mathbf{r}_1|$ is the distance between objects 1 and 2, and $\hat{\mathbf{r}}_{12} \stackrel{\text{def}}{=} \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|}$ is the unit vector from object 1 to 2.

2.5 Problems with Newton's Theory

This description of gravity is sufficiently accurate for man practical purposes and is widely used. Deviations from it are small when the dimensionless quantities φ/c^2 and $(v/c)^2$ are both much less than one, where φ is the gravitational potential, v is the velocity of the objects being studied, and c is the speed of light. Newtonian gravity provides an accurate description of the Earth/Sun system:

$$\frac{\Phi}{c^2} = \frac{GM_{\text{sun}}}{r_{\text{orbit}}c^2} \sim 10^{-8}, \quad \left(\frac{v_{\text{Earth}}}{c}\right)^2 = \left(\frac{2\pi r_{\text{orbit}}}{(1\ \text{yr})c}\right)^2 \sim 10^{-8}$$

where r_{orbit} is the radius of the Earth's orbit around the sun. In situations where either dimensionless parameter is large, general relativity must be used to describe the system. General relativity reduces Newtonian gravity in the limit of small potential and low velocities, so Newton's aw of gravitation can also be called the low-gravity limit of general relativity.

3. General Relativity

General relativity is the geometric theory of gravitation published by the great scientist Albert Einstein in 1916. General relativity states that gravitational forces of matter and masses cause space to curve. Before general relativity was found, Newton's law of universal gravitation had been used for more than 200 years to describe gravitation between masses. Einstein's general theory of general relativity has helped to account for several effects that have not been explained by Newton's law, such as minute anomalies in the orbits of Mercury and other planets. General relativity also predicts effects of gravity such as gravitational waves, is a fluctuation in the curvature of spacetime which propagates as a wave; gravitational lensing, which is when light from a distant, bright star is bent around a massive object; and gravitational time dilation, the effect of time passing at different rates in areas of different gravitational potential.

General relativity has been an essential tool in modern astrophysics, providing the foundation for our understanding of black holes today. General relativity has proven to be consistent with experimental data.

3.1 Einstein's Equation

In this section, we would be looking at Einstein's equations which show the relationship between spacetime geometry and the properties of matter. They are formulated using concepts of Riemannian geometry, where the geometric properties of a spacetime are described by a metric. The metric encodes the data needed to compute the fundamental geometric notions of distance and angle in a spacetime. The location of any point on any surface can be described by coordinates, but coordinates do not provide enough information to describe the geometry of a spherical surface. This information is encoded in the metric, which is a function that is defined at every point on the surface or spacetime and relates the coordinate differences to differences in distance. Other such as the length of any given curve or the angle at which two curves meet can be computed from this metric function.

This metric function and its rate of change from point to point are able to define a geometrical quantity called the Riemann curvature tensor, which describes how the space is curved at each point. In general relativity, the metric and the Riemann curvature tensor are quantities that are defined at every point in spacetime. The matter content of the spacetime defines another quantity, the energy-momentum tensor T, and the principal that states "spacetime tells matter how to move, and matter tells spacetime how to curve" means these quantities must be related in some formula. Einstein has represented this relationship in a formula by using the Riemann curvature tensor and the metric to define another geometrical quantity G, the Einstein tensor, which describes some aspects of the way spacetime is curved. Einstein's equation states:

$$\mathbf{G} = \frac{8\pi G}{c^4}\mathbf{T},$$

where G is the Einstein tensor, *G* is the gravitational constant, c is the speed of light and T is the energy-momentum tensor. The quantity G, which measures curvature, is equated with T, which measures matter content. The constants in this equation reflect the theories that went into its making – *G* the gravitational constant in Newtonian gravity, *c* the speed of light used in special relativity, and π one of the basic constants of geometry.

4. Black Holes

4.1 Introduction to Black Holes

We have decided to look at black holes as they are great massive objects in space with exceptionally strong gravitational fields and they exhibit exotic phenomena. A black hole is a body that has such a great mass and gravitational field that nothing, not even light, can escape from it because of its gravitation. There is a one-way surface in a black hole, into which objects can fall but cannot escape. At this surface, the Schwarzschild radius, in the theory of relativity, time stops. Because nothing is able to escape from beyond the Schwarzschild radius, it is also known as the event horizon. A star may end off as a black hole, as a black hole is created when a huge star has burned all of its fuel and the heat can no longer withstand the large pressure of gravitation.

4.2 Escape Velocity and Newton's Law

A black hole is also described as an object that has an escape velocity greater than the speed of light. The escape velocity of an object is the speed it needs to travel at so that it would be able to escape a source of gravity without falling back into orbit around it. To find the escape velocity in Newtonian mechanics, we will consider a heavy object with mass *M* centered at the origin. A second object with mass m starting at a distance *r* from the origin with a speed *v*, trying to escape gravity, needs just enough energy to make up for the negative gravitational potential energy. It would have no energy left over, as seen in the equation below:

$$\frac{mv^2}{2} - \frac{GMm}{r} = 0$$,

where the subtraction of the gravitational potential energy from the kinetic energy gives 0. As *r* gets closer to infinity, it would have less and less kinetic energy and finally end up at infinity with no speed. Hence, we can find the critical escape velocity *v* in terms of *M* and *r*.

It is also said that for each value of *v* and *M*, there is a critical value for *r* such that an object with speed v is just able to escape, given by the formula:

$$r = \frac{2GM}{v^2}$$.

When the velocity is equal to the speed of light, this gives the radius of a Newtonian dark star, which is theoretically a body from which a particle moving at the speed of light is unable to escape. The value of the radius of a black hole, given by the radius of the event horizon, is often taken as being equal to this Newtonian value:

$$r_{\text{Schwarzschild}} = \frac{2GM}{c^2}$$

where *c* is the speed of light. The escape velocity of an object depends on how compact the object is, given by the ratio of its mass to its radius. A black hole forms when an object is so compact that even having the speed of light is not fast enough to escape.

4.3 The No Hair Theorem

The no hair theorem, in astrophysics, states that once a black hole settles down, it can be characterised completely by its mass, angular momentum, and electric charge. Angular momentum is related to particles moving around a point, and all moving objects have angular momentum, but angular momentum is mostly used to describe rotating objects.

4.3.1 Naming Of The Theorem

There had always been questions on the shape of black holes formed when objects collapse. Would objects of different shapes, for example sphere or cube, collapse into different shaped black holes? However, it turned out that all black holes would collapse to form simple black holes, not at all retaining their original shapes. This amazing property of black holes had been popularized by physicist John Wheeler as the no hair theorem as he likened it to how black holes had were indistinguishable, having similar "hair styles", hence the name "no hair theorem".

4.3.2 The Theorem

Any two black holes that share the same values for mass, angular momentum and charge, or parameters, are indistinguishable. These parameters are special as they can be found and derived outside of the black hole. The mass and electric charge can be found with the help of Gauss's law, which states that the electrical flux through any closed surface is proportional to the enclosed electric charge. Likewise, the angular momentum can be measured using frame dragging by the gravitomagnetic field. Frame dragging would be described in the Kerr metric in the next section.

4.4 The Kerr Solution

In general relativity, the Kerr metric, discovered by New Zealand mathematician Roy Kerr, describes the geometry of spacetime around a rotating massive body. According to the Kerr metric, such rotating bodies would exercise frame dragging, where the bodies would drag spacetime around themselves. The Kerr metric is often used on rotating black holes, which exhibit great phenomena. Such black holes have two surfaces where the metric seems to have a singularity. The shape and the size of these surfaces depend on its mass and angular momentum. The inner surface is the event horizon and the outer surface encloses the ergosphere. Any object passing between these two horizons must co-rotate with the rotating body. This feature can be used to extract energy from the black hole, up to its invariant mass, mc^2. The Kerr metric describes the geometry of spacetime in the vicinity of mass M rotating with angular momentum J, and the mathematical formula is given below:

$$c^2 d\tau^2 = \left(1 - \frac{r_s r}{\rho^2}\right) c^2 dt^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 - \left(r^2 + \alpha^2 + \frac{r_s r \alpha^2}{\rho^2} \sin^2\theta\right) \sin^2\theta \; d\phi^2 + \frac{2 r_s r \alpha \sin^2\theta}{\rho^2} c\, dt\, d\phi$$

where coordinates r, θ, φ are standard spherical coordinate system and r_s is the Schwarzschild radius.

5. Conclusion

Our group has managed to look at many laws and scientific theories from many great scientists and indeed, mathematics has played a great role in helping us derive answers and to find values that had once been impossible to. For example, we are able to calculate the force of gravity between two planets by using Newton's law of universal gravitation, and calculate the escape velocity of particles. The world of astronomy has been brought closer to us with the help of mathematics, and we hope that further discoveries by scientists and mathematicians would continue to bring us greater understanding of astronomy.

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